Stability Analysis of Fractional Order Two Species Interactions

A. George Maria Selvam[#1], D.Vignesh[#2]

[#]*Department of Mathematics, Sacred Heart College (Autonomous),*
Tirupattur, Vellore Dist., Tamil Nadu, S.India.
[1]agmshc@gmail.com
[2]dvignesh260@gmail.com

***Abstract*— The prey predator modeling attempts to analyse the factors that involve in the existence of the species sharing the same habitat. Long range memory of many biological systems is very well modeled by system of fractional order differential/difference equations. This work considers a two species interaction described by fractional order model. Existence and uniqueness of solution for the fractional order system is established and the stability of the system for the commensurate and incommensurate fractional order is investigated. Numerical simulations for different parameter values are presented to study the coexistence of the species. The lyapunov characteristics exponent are tabulated with simulation.**

***Keywords*— Fractional differential equations, Prey predator model, Fixed points, Stability, Lyapunov Exponent.**

I. Introduction

Many real life phenomena are studied using fractional calculus, which is the generalization of the integer order differential equations. ([8]) Nowadays, fractional calculus has attained much attention as the novel mathematical model to describe various phenomena in nature. Due to the non-local property of the fractional derivative, the fractional order differential equations are much preferred than ordinary differential equations in biological, economic and social systems where memory effects are important.

Mathematical Biology is the branch of mathematics that study, analyse and interpret biological phenomena such as interaction, evolution and co-existence of species. Interaction here is not only among the individuals of same species but also among individuals of different species, disease, environmental factors and supply of food. ([1],[2]) Prey-predator interaction has been most important topic in Mathematical modeling. The history of the predator-prey modeling dates back to year 1910 when Autocatalytic reactions was modeled using logistic equations by Alfred. J. Lotka which was originally given by Pierre Francois Verhulst. In 1920, he extended the model for plants and animal species.

Modeling of prey predator interactions using the same set of equations was given by Volterra in the year 1926, when he modeled the evolution of the fish population in Adriatic sea. ([4],[6]) Later the model was further developed considering various factors that affect the interaction of the two species like density dependent growth. The model with limited food resources for the predator was developed by C.S Holling using functional responses. At the end of 20th century ratio dependent models began to emerge.

In this work, the two species interaction model with fractional order is considered. This paper is organized as follows. Section 2 contains preliminaries and the mathematical model and existence and uniqueness of the system is discussed in Section 3. The stability of the system with numerical simulations is carried out in Section 4 with Lyapunov Exponent in Section 5 and the conclusion in Section 6.

II. Preliminaries

In this section, we provide some basic definitions and theorems to study the stability of the fractional order systems. Let

$$\begin{aligned} D^{\alpha_1}u_1(t) &= g_1(u_1,u_2,\dots,u_n) \\ D^{\alpha_2}u_2(t) &= g_1(u_1,u_2,\dots,u_n) \\ &\vdots \\ D^{\alpha_n}u_n(t) &= g_n(u_1,u_2,\dots,u_n) \end{aligned} \qquad (1)$$

with initial values

$$u_1(0) = u_{01}, u_2(0) = u_{02}, \dots, u_n(0) = u_{0n}$$

represent the general form of the Fractional derivative system with $0 < \alpha_i \le 1, i = 1, 2, 3...$ being the fractional order of the system.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the system (1) is called commensurate order fractional derivative system. Otherwise it is called an incommensurate order fractional derivative system.

Theorem 1 *([3], [5])*

Consider the commensurate nonlinear fractional-order system (1) with its order $0 < \alpha_i \le 1$. The equilibrium points u^* of system (1), are locally asymptotically stable if all eigenvalues μ_i of the Jacobian Matrix J at the equilibrium points

$$J = [a_{ij}], \; i, j = 1, 2, \dots, n$$

where

$$a_{ij} = \frac{\partial f_i}{\partial u_j} |u^*$$

satisfy

$$|arg(\mu_i)| > \frac{\alpha\pi}{2}$$

Theorem 2 *[7]*

Consider the incommensurate nonlinear fractional-order system (1) with its order is ratio of rational numbers between $0 < \alpha_i \le 1$. Let M be the LCM of the denominators c_i of a_{ij} where $\alpha_i = \frac{q_i}{c_i}, GCD(q_i, c_i) = 1, q_i, c_i \in \mathbf{Z}^+, i = 1, 2, 3...$ and $\gamma = \frac{1}{M}$. Then the equilibrium points u^* of system (1) are locally asymptotically stable iff all roots μ_i of the equation

$$det \begin{pmatrix} \mu^{M\alpha_1} - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \mu^{M\alpha_2} - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \mu^{M\alpha_n} - a_{nn} \end{pmatrix} = 0 \qquad (2)$$

satisfy

$$|arg(\mu_i)| > \frac{\gamma\pi}{2}.$$

III. MATHEMATICAL MODEL

The system

$$\frac{d^\alpha u(t)}{dt^\alpha} = r(u(t) - u(t)^2) - bu(t)v(t) - eu(t)$$

$$\frac{d^\alpha v(t)}{dt^\alpha} = -s(v(t) + v(t)^2) + du(t)v(t) \qquad (3)$$

where α represents the fractional order and $u(t)$, $v(t)$ denote the population of prey and predator respectively. Growth rate of prey is represented by r and b is the consumption rate of prey by predator.

The decline in the population of predator is s, increase in population of Prey due to interaction with prey is d and e is the natural death of the prey. All the parameter are assumed to take positive values.

A. *Existence and Uniqueness*

Theorem 1 The sufficient condition for existence and uniqueness of the solutions of the fractional-order system (3) in the region $\Omega \times (0,T]$ with initial conditions $W(0) = W_0$ and $t \in (0,T]$

$$Q = \frac{T^{\alpha}}{\Gamma(\alpha+1)} max\{\{a(1+2\beta)+(b+d)\beta+e\},\{c(1+2\beta)+(b+d)\beta\}\} < 1$$

Proof.

The Existence and uniqueness of the solution is studied in the region $\Omega \times (0,T]$ where $\Omega = \{(u,v) \in \mathbb{R}_+^2 : \max(|u|,|v| \le \beta)\}$ based on contraction mapping principle. The fractional derivative system (3) is written as

$$D^{\alpha}W(t) = G(W(t)), t \in (0,T], W(0) = W_0$$

where

$$W(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, W_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix},$$

$$G(W) = \begin{bmatrix} r(u(t)-u(t)^2) - bu(t)v(t) - eu(t) \\ -s(v(t)+v(t)^2) + du(t)v(t) \end{bmatrix}$$

The maximum norm is defined as

$$\|L\| = \max_{t\in(0,T]} |L(t)|$$

The following definition of the Norm of a Matrix $A = [a_{ij}(t)]$ is used in the following discussion

$$\|A\| = \max_{j} \sum_i |a_{ij}|$$

The Solution of the Fractional derivative system (3) with is

$$W(t) = W_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} G(W(s))ds = P(W)$$

So,

$$P(W_1) - P(W_2) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(G(W_1(s)) - G(W_2(s)))ds$$

From above, one can obtain the inequality as

$$|P(W_1) - P(W_2)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|G(W_1(s)) - G(W_2(s))|ds$$

whichimplies

$$\|P(W_1) - P(W_2)\| \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} max\{\{a(1+2\beta)+(b+d)\beta+e\},\{c(1+2\beta)+(b+d)\beta\}\}\|W_1 - W_2\|$$

$$\le Q\|W_1 - W_2\|$$

where

$$Q = \frac{T^{\alpha}}{\Gamma(\alpha+1)} max\{\{a(1+2\beta)+(b+d)\beta+e\},\{c(1+2\beta)+(b+d)\beta\}\}$$

Thus $P(W)$ satisfies the Lipschitz condition. Clearly the mapping $W = P(W)$ is a contraction mapping if $Q < 1$. Hence the theorem.

IV. Stability Analysis

In this section, we discuss the stability of the commensurate and incommensurate order fractional derivative system by using Theorems 1 and 2.

The equilibrium points of the system (3) are

1. $E_0(u,v) = (0,0)$
2. $E_1(u,v) = \left(\frac{r-e}{s}, 0\right)$
3. $E_2(u,v) = \left(\frac{s(b-e+r)}{bd+rs}, \frac{dr+de-rs}{bd+rs}\right)$

In this work, we discuss the stability of the prey predator system for the interior equilibrium point (E_2).

A. *Commensurate Fractional Derivative System*

Consider the fractional derivative system (2) with commensurate order. The Jacobian matrix of the system (3) at (u^*, v^*) is

$$J(u^*,v^*) = \begin{bmatrix} r(1-2u^*) - bv^* - e & -bu^* \\ -dv^* & -s(1+2v^*) \end{bmatrix}$$

The Jacobian Matrix at E_2 is

$$J(E_2) = \begin{bmatrix} r - \left(\frac{2sr(b-e+r)}{bd+sr}\right) - \frac{b(-de+dr-sr)}{bd+sr} - e & -\frac{bs(b-e+r)}{bd+sr} \\ \frac{(-de+dr-sr)d}{bd+sr} & -s - \left(\frac{s(-de+dr-sr)}{bd+sr}\right) + \frac{ds(b-e+r)}{bd+sr} \end{bmatrix}$$

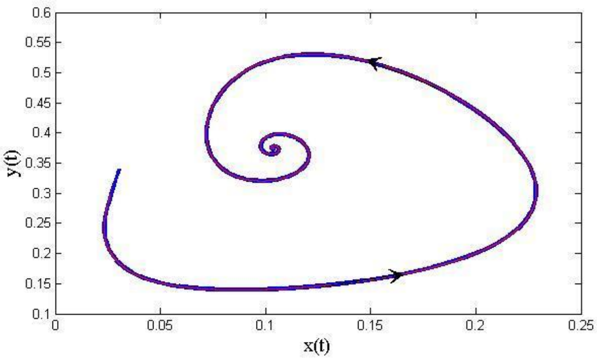
1) *Numerical Examples*

Example 1

Considering the parametric values $r = 0.12, b = 0.26, s = 0.03, d = 0.4, e = 0.01$ *and* $\alpha = 0.95$ *with initial conditions* $u(0) = 0.03, v(0) = 0.34$ *the eigenvalues of the system (3) are*

$$\mu_{1,2} = -0.01182 \pm 0.06346\ i$$

Clearly, $|arg\,\mu_{1,2}| > \frac{\alpha\pi}{2}$, thus the system (3) is asymptotically stable as in Figure-1.



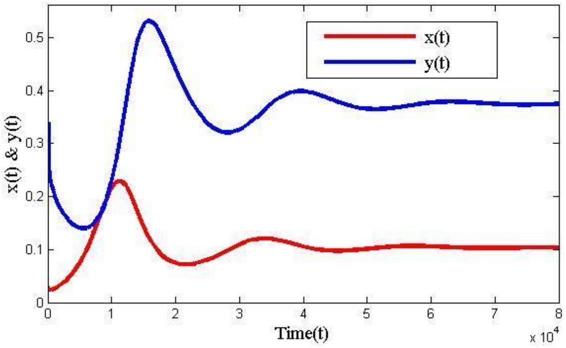


Figure 1: Phase trajectory and Time plots of the system (3).

Time plot of the prey and predator species in Figure-1 illustrates the gradually growing predator population with increase in population of the prey. Simultaneously decreasing prey population results in decline in predation and predator's population density. The coexistence of the prey and predator is attained with increase in time. Phase portait starting from the initial points forms a spiral curve moving in towards the equilibrium point ensuring the asymptotic stability of the system.

B. *Incommensurate Fractional Order System*

For Incommensurate order , the system (3) takes the form

$$\frac{d^{\alpha_1}u(t)}{dt^{\alpha_1}} = r(u(t) - u(t)^2) - bu(t)v(t) - eu(t)$$

$$\frac{d^{\alpha_2}v(t)}{dt^{\alpha_2}} = -s(v(t) + v(t)^2) + du(t)v(t) \qquad (4)$$

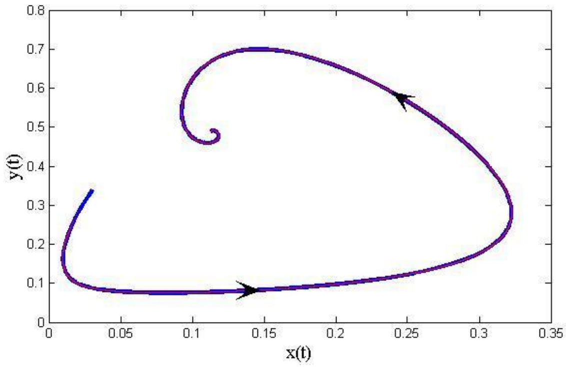
where α_1, α_2 is the fractional order and $\alpha_1 \neq \alpha_2$. The stability analysis of the fractional derivative system with incommensurate order is carried out with the help of roots of the higher order polynomial function obtained by transformation of system (4) using (2). The order of the polynomial function depends on the fractional order considered for the system (4). The condition in Theorem- (2) is to be checked for all the roots of the polynomial function.

Here in this section we provide two numerical examples of the system with incommensurate fractional order.

1) *Numerical Examples*

Example 2

For the system (4) with order $\alpha_1 = 0.8, \alpha_2 = 0.9$ *and the values of the parameters* $r = 0.162, b = 0.27, s = 0.03, d = 0.4, e = 0.01$, *the minimum value of the argument of the roots of the characteristic equation obtained using (2) is* $min|arg(\mu)| = 0.369662$. *The value of* $\frac{\gamma\pi}{2} = 0.157142$, *where* $\gamma = \frac{1}{M}$. Since $|arg(\mu_i)| > \frac{\gamma\pi}{2}$, *the system (4) is asymptotically stable. Figure-2 illustrates the stability of the equilibrium point of the system (4) with initial condition* $u(0) = 0.03, v(0) = 0.34$.



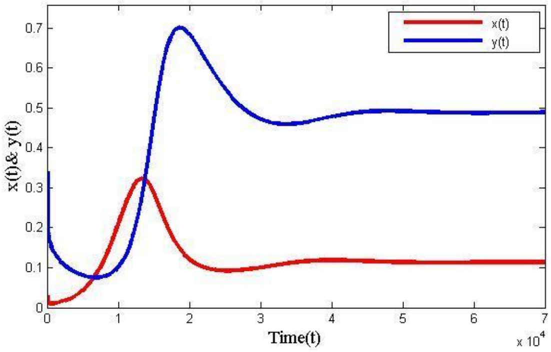


Figure 2: Phase trajectory and Time plots of the system (3).

The time plots from the Figure-2 explains the fact that initially the growth the predator stagnates as the growth of the prey is small when compared to the predation rate. Once the prey population reaches its maximum population level it stimulates the growth of the predator. Over the period of time there is no variation in the population density of both prey and predator. Thus the stability between the species population is thus attained.

V. LYAPUNOV EXPONENT

In dynamical systems, the exponential growth or decay of phase space perturbations are measured using the lyapunov exponents. Lyapunov characteristic exponent gives the separation rate of the infinitesimally close trajectories.

The values of the lyapunov exponent of the system with $r=0.2, b=0.25, s=0.01, d=0.54, e=0.01$ and fractional order $\alpha=0.9$ are tabulated in Table-1 and plotted in Figure-3. For $t\in[0,300]$, the lyapunov exponent of the system (3) is $(-0.0105,-0.0114)$.

TABLE 1
TIME EVOLUTION OF LE

Time	LE1	LE2	Time	LE1	LE2
10	0.2019	0.0303	160	-0.0098	-0.0083
20	0.0035	-0.0360	170	-0.0094	-0.0099
30	0.0049	-0.1185	180	-0.0207	-0.0020
40	-0.0064	-0.1083	190	-0.0118	-0.0130
50	-0.0124	-0.0820	200	-0.0110	-0.0137
60	-0.0156	-0.0535	210	-0.0128	-0.0104
70	-0.0182	-0.0261	220	-0.0167	-0.0048
80	-0.0252	0.0028	230	-0.0102	-0.0106
90	-0.0046	-0.0037	240	-0.0110	-0.0104
100	-0.0065	-0.0073	250	-0.0177	-0.0048
110	-0.0156	-0.0110	260	-0.0114	-0.0118
120	-0.0112	-0.0210	270	-0.0109	-0.0122
130	-0.0115	-0.0198	280	-0.0135	-0.0090
140	-0.0132	-0.0136	290	-0.0131	-0.0089
150	-0.0208	-0.0007	300	-0.0105	-0.0114

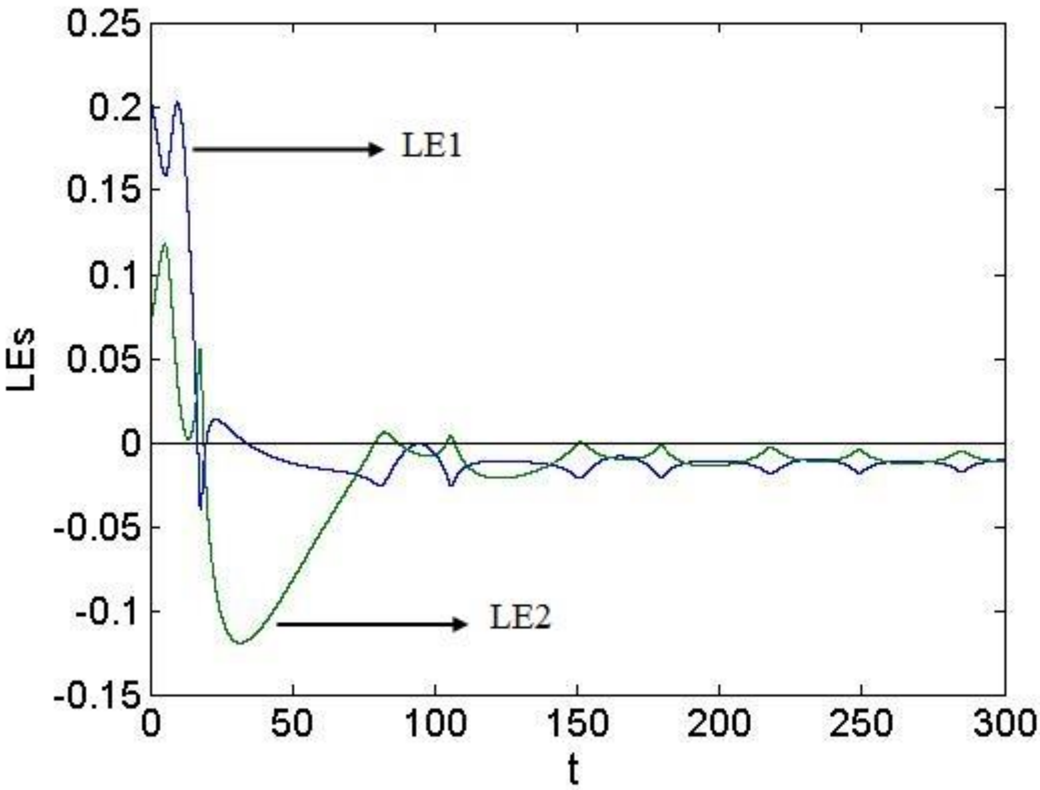


Figure 3: Lyapunov Exponent of the system (3).

VI. CONCLUSIONS

In this work, the existence and uniqueness of the solution for the fractional derivative prey predator system is established. The stability of the system for commensurate order is studied with eigenvalues obtained from the jacobian matrix of the system. The stability of system with incommensurate order is dealt by finding the characteristic equation and illustrated with numerical examples. Phase portrait and time plots are provided supporting theoretical study. The *lyapunov* exponents of the system are tabulated and simulation is performed.

REFERENCES

[1] A. E. Matouk and A. A. Elsadany, "Dynamical analysis, stabilization and discretization of a chaotic fractional-order GLV model",Nonlin. Dyn. 85, 1597–1612 (2016).
[2] El-Saka, H & Lee, Seyeon & Jang, Bongsoo. (2019), "Dynamic analysis of fractional-order predator–prey biological economic system with Holling type II functional response",Nonlinear Dyn. 10.1007/s11071-019-04796-y.
[3] El-Saka, H.A.,El-Sayed, A.: "Fractional Order Equations and Dynamical Systems" Lambert Academic Publishing, Saarbrucken.ISBN 978-3-659-40197-8 (2013).
[4] A. George Maria Selvam, R.Janagaraj and P. Rathinavel, "Stability in a Fractional Order Three Species Interaction Model", International Journal of Engineering Research & Management Technology , ISSN : 2348-4039, Volume 2, Issue-4, pp 11-18 (July 2015).
[5] Ahmed, E., El-Sayed, A.M.A., El-Saka, H.A.A., "Equilibrium points, stability and numerical solutions of fractional order predator–prey and rabies models", J.Math. Anal. Appl.325, 542–553 (2007).
[6] A. George Maria Selvam, R.Janagaraj and D. Abraham Vianny, "Dynamics In A Fractional Order Prey - Predator Interactions", Mathematical Modelling and Applied Computing,ISSN 0973-6093, Volume 6, Number 1 (2015), pp. 1-6.
[7] Deng, W., Li, C., Lu, J.: "Stability analysis of linear fractional differential system with multiple time delays". Nonlinear Dyn. 48, 409–416 (2007).
[8] A. George Maria Selvam, R. Dhineshbabu and D. Abraham Vianny, "Analysis of a Fractional Order Prey Predator Model(3-species)", Global Journal of Computational Science and Mathematics, ISSN 2248-9908, Volume 5, Number 1 (2016), pp.1-9.