Fixed Point and Common Fixed Point Theorems on Complex Valued b-metric spaces

[1] D.Dayana Roselin, [2] J.Carmel Pushpa Raj [3] J.Maria Joseph

[1] Department of Mathematics,
Auxilium College of Arts and Science for Women, Pudukkottai - 614 602, India.
[2,3] P.G. and Research Department of Mathematics,
St.Joseph's College (Autonomous), Tiruchirappalli - 620 002, India.
[1] jose.rose80@gmail.com, [2] carmelsjc@gmail.com, [3] joseph80john@gmail.com,

Abstract

In this Chapter, We discuss the study of fixed Point and common fixed point theorems in complex valued b-metric spaces which generalize and unify some known results in the current literature.
Keywords : Common fixed point,Rational Contraction, Complex valued b-metric space.

1 Introduction

In 1922,Banach proved a contraction principle.This Banach contraction [1] mapping principle was one of the important theorems for fixed point theorems. Bakhtin [2] introduced the notion of b-metric space which generalise the metric space. In [5], Czerwik proved the contraction mapping principle in b-metric spaces.Continuously many authors worked on single valued and multivalued mapping and obtaied the fixed point. A new space called the complex valued metric space was introduced by Azam et ai. [7],he is the one to involve rational inequality. Many authors studied many common fixed point theorems on Complex valued metric space. The concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [3]. In Sequel, Mukheimer [6] proved some common fixed point theorems in complex valued b-metric spaces

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers abd $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. Thus $z_1 \precsim z_2$ if one of the following holds:

$$(1) Re(z_1) = Re(z_2) \text{and} Im(z_1) = Im(z_2);$$
$$(2) Re(z_1) < Re(z_2) \text{and} Im(z_1) = Im(z_2);$$
$$(3) Re(z_1) = Re(z_2) \text{and} Im(z_1) < Im(z_2);$$
$$(4) Re(z_1) < Re(z_2) \text{and} Im(z_1) < Im(z_2);$$

We write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.
It follows that
(i) $0 \preceq z_1 \preceq z_2 \text{implies} |z_1| < |z_2|$;

(ii) $z_1 \preceq z_2$ and $z_2 \prec z_3$imply $z_1 \prec z_3$;
(iii) $0 \preceq z_1 \preceq z_2$ implies $|z_1| \leq |z_2|$;
(iv) if$a, b \in \mathbb{R}, 0 \leq a \leq bandz_1 \preceq z_2$, then $az_1 \preceq bz_2$for all $z_1, z_2 \in \mathbb{C}$ Recently Rao et al.[3] introduced Recently Rao et al.[3] introduced the following definition.

Definition 2.1. *Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $a, b, c \in X$ the following conditions are satisfied:*

1. $0 \precsim d(a, d)$*and*$d(a, b) = 0$ *if and only if* $a = b$ *for all* $a, b \in X$;
2. $d(a, b) = d(b, a)$ *for all* $a, b \in X$;
3. $d(a, b) \precsim s[d(a, c) + d(c, b)]$ *for all* $a, b, c \in X$.

Then (X, d) is called a complex valued $b-$metric space.

Example 2.2. *[3] If $X = [0, 1]$, define the mapping $d : X \times X \to \mathbb{C}$ by $d(a, b) = |a-b|^2 + i|a-b|^2$, for all $a, b \in X$. Then (X, d) is complex valued b-metric space with $s = 2$.*

Definition 2.3. *[3] Let (X, d) be a complex valued b-metric space*
(i) A point $a \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(a, r) = \{b \in X : d(a, b) \prec r\} \subseteq A$.
(ii) A point $a \in X$ is called limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$,
$B(a, r) \cap (A - a) \neq \phi$
(iii)A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.
(iv) A subbasis for a Hausdorff topology τ on X is a family
$F = \{B(a, r) : x \in X \text{and} 0 < r\}$

Definition 2.4. *[3] Let (X, d) be a complex valued b-metric space and $\{a_n\}$ be a sequence in X and $a \in X$.*
(i) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that for all $n > N, d(a_n, a) \prec c$, then $\{a_n\}$ is said to be convergent,$\{a_n\}$ converges to a, and a is the limit point of $\{a_n\}$.It is denoted by $\lim_{n\to\infty} a_n = a$ or $\{a_n\} \to a$ as $n \to \infty$.
(ii) If for every $c \in \mathbb{C}$ with $0 \prec r$,, there is $N \in \mathbb{N}$ such that for all $n > N, d(a_n, a_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{a_n\}$is said to be Cauchy sequence.
(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 2.5. *[3] Let (X, d) be a complex valued b-metric space and let $\{a_n\}$ be a sequence in X. Then $\{a_n\}$ converges to a if and only if $|d(a_n, a)| \to 0$ as $n \to \infty$.*

Lemma 2.6. *[3] Let (X, d) be a complex valued b-metric space and let $\{a_n\}$ be a sequence in X. Then $\{a_n\}$ is a Cauchy sequence if and only if $|d(a_n, a_{n+m})| \to 0$ as $n \to \infty$. where $m, n \in \mathbb{N}$.*

3 Main Reults

Theorem 3.1. *Let* (X, d) *be a complete complex valued b-metric space with* $s \geq 1$ *and let* S,T *be self-mappings from X into itself satisfy the following inequality,*

$$d(Sa, Tb) \leq a_1 d(a, b) + a_2[\frac{d(a, Sa)d(b, Tb)}{d(a, b) + d(a, Tb) + d(b, Sa)}] + a_3[\frac{d(a, Ta)d(b, Tb)[1 + d(b, Sa)d(a, b)]}{d(a, Tb) + d(a, b)}] \quad (1)$$

for all $a, b \in X$ *such that* $a \neq b, d(a, b) + d(a, Tb) + d(b, Sa) \neq 0, d(a.Tb) + d(a, b) \neq 0$ *where* a_1, a_2*and* a_3 *are nonnegative reals with* $a_1 + s(a_2 + a_3) \prec 1$ *or* $d(Sa, Tb) = 0$ *if* $d(a, Tb) + d(b, Sa) + d(a, b) = 0, d(a, Tb) + d(a, b) = 0.$
Then S and T have a unique common fixed point.

Proof : For any arbitrary point $a_0 \in X$, define sequence $\{a_n\}$ in X such that

$$a_{2n+1} = Sa_{2n},$$
$$a_{2n+2} = Ta_{2n+1}, \forall n \geq 0.$$

Now,we show that the sequence $\{a_n\}$ is Cauchy sequence.
Let $a = a_{2n}$ and $b = a_{2n+1}$ in 3.1; we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(Sa_{2n}, Ta_{2n+1}) \\ &\leq a_1 d(a_{2n}, a_{2n+1}) \\ &\quad + a_2[\frac{d(a_{2n}, Sa_{2n})d(a_{2n+1}, Ta_{2n+1})}{d(a_{2n}, Ta_{2n+1}) + d(a_{an+1}, Sa_{2n}) + d(a_{2n}, a_{2n+1})}] \\ &\quad + a_3[\frac{d(a_{2n}, Ta_{2n})d(a_{2n+1}, Ta_{2n+1})[1 + d(a_{2n+1}, Sa_{2n})d(a_{2n}, a_{2n+1})]}{d(a_{2n}, Ta_{2n+1}) + d(a_{2n}, a_{2n+1})}] \\ &\leq a_1 d(a_{2n}, a_{2n+1}) \\ &\quad + a_2[\frac{d(a_{2n}, a_{2n+1})d(a_{an+1}, a_{2n+2})}{d(a_{2n}, a_{2n+2}) + d(a_{an+1}, a_{2n+1}) + d(a_{2n}, a_{2n+1})}] \\ &\quad + a_3[\frac{d(a_{2n}, a_{2n+1})d(a_{an+1}, a_{2n+2})[1 + d(a_{2n+1}, a_{2n+1})d(a_{an}, a_{2n+1})]}{d(a_{2n}, a_{2n+2}) + d(a_{2n}, a_{2n+1})}] \\ |d(a_{2n+1}, a_{2n+2})| &\leq a_1|d(a_{2n}, a_{2n+1})| \\ &\quad + a_2[\frac{|d(a_{2n}, a_{2n+1})||d(a_{an+1}, a_{2n+2})|}{|d(a_{2n}, a_{2n+2})| + |d(a_{2n}, a_{2n+1})|}] \\ &\quad + a_3[\frac{|d(a_{2n}, a_{2n+1})||d(a_{an+1}, a_{2n+2})|}{|d(a_{2n}, a_{2n+2})| + |d(a_{2n}, a_{2n+1})|}] \\ &\leq a_1|d(a_{2n}, a_{2n+1})| \\ &\quad + a_2[\frac{|d(a_{2n}, a_{2n+1})|s[|d(a_{an+1}, a_{2n})| + |d(a_{an}, a_{2n+2})|]}{|d(a_{2n}, a_{2n+2})| + |d(a_{2n}, a_{2n+1})|}] \\ &\quad + a_3[\frac{|d(a_{2n}, a_{2n+1})|s[|d(a_{an+1}, a_{2n})| + |d(a_{an}, a_{2n+2})|]}{|d(a_{2n}, a_{2n+2})| + |d(a_{2n}, a_{2n+1})|}] \end{aligned}$$

$|d(a_{2n+1}, a_{2n+2})| \leq (a_1 + s(a_2 + a_3))|d(a_{2n} + a_{2n+1})|$
$\Longrightarrow |d(a_{2n+1}, a_{2n+2})| \leq \lambda|d(a_{2n} + a_{2n+1})|$
where$\lambda = (a_1 + s(a_2 + a_3)) \prec 1$

3

Similarly,we obtain $|d(a_{2n+2}, a_{2n+3})| \leq \lambda|d(a_{2n+1} + a_{2n+2})|$
Hence,with $\lambda = (a_1 + s(a_2 + a_3)) \prec 1$,and for all $n \geq 0$,and consequently,we have

$$\begin{aligned}|d(a_{2n+1}, a_{2n+2})| &\leq \lambda|d(a_{2n}, a_{2n+1})| \\ &\leq \lambda^2|d(a_{2n-1}, a_{2n})| \leq \cdots \\ &\leq \lambda^{2n+1}|d(a_0, a_1)|\end{aligned}$$

That is,for all $n \in \mathbb{N}$,we can write

$$\begin{aligned}|d(a_{n+1}, a_{n+2})| \leq \lambda|d(a_n, a_{n+1})| \leq \lambda^2|d(a_{n-1}, a_n)| \\ \leq \cdots \leq \lambda^{n+1}|d(a_0, a_1)|\end{aligned} \tag{2}$$

Fix $m > n; m, n \in N$, we have

$$\begin{aligned}|d(a_n, a_m)| &\leq s|d(a_n, a_{n+1})| + s|d(a_{n+1}, a_m)| \\ &\leq s|d(a_n, a_{n+1})| + s^2|d(a_{n+1}, a_{n+2})| + s^2|d(a_{n+2}, a_m)| \\ &\leq \cdots \\ &\leq s|d(a_n, a_{n+1})| + s^2|d(a_{n+1}, a_{n+2})| + s^3|d(a_{n+2}, a_{n+3})| \\ &\quad + \cdots + s^{m-n-1}|d(a_{m-2}, a_{m-1})| + s^{m-n}|d(a_{m-1}, a_m)|.\end{aligned}$$

By using ,we get

$$\begin{aligned}|d(a_n, a_m)| &\leq s\lambda^n|d(a_0, a_1)| + s^2\lambda^{n+1}|d(a_0, a_1)| + s^3\lambda^{n+2}|d(a_0, a_1)| \\ &\leq \cdots \leq +s^{m-n-1}\lambda^{m-2}|d(a_0, a_1)| + s^{m-n}\lambda^{m-1}|d(a_0, a_1)| \\ &\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}]|d(a_0, a_1)| \\ &\leq s\lambda^n[1 + s\lambda + s^2\lambda^2 + \cdots + s^{m-n-1}\lambda^{m-n-1}]|d(a_0, a_1)| \\ &= s\lambda^n \sum_{k=0}^{\infty}(s\lambda)^k|(d(a_0, a_1))| \\ &\leq \frac{s\lambda^n}{1 - s\lambda}|d(a_0, a_1)|\end{aligned}$$

Therefore, $|d(a_n, a_m)| \leq \dfrac{s\lambda^n}{1 - s\lambda}|d(a_0, a_1)| \to 0$ as $n \to \infty$
Thus, $\{a_n\}$ is a Cauchy sequence in X.Since X is complete,there exists some $v \in X$ such that $a_n \to v$ as $n \to \infty$.Assume not,then there exists $z \in X$ such that

$$|d(v, Sv)| = |z| > 0 \tag{3}$$

So by using the triangular inequality and 1,we get

$$\begin{aligned}z &= d(v, Sv) \\ &\leq sd(v, a_{2n+2}) + sd(a_{2n+2}, Sv) \\ &= sd(v, a_{2n+2}) + sd(Ta_{2n+1}, Sv) \\ &\leq sd(v, a_{2n+2}) + sa_1d(v, a_{2n+1}) + sa_2[\frac{d(v, Sv)d(a_{2n+1}, Ta_{2n+1})}{d(v, Ta_{2n+1}) + d(a_{2n+1}, Sv) + d(v, a_{2n+1})}] \\ &\quad + sa_3[\frac{d(v, Tv)d(a_{2n+1}, Ta_{2n+1})[1 + d(a_{2n+1}, Sv)d(v, a_{2n+1})]}{d(v, Ta_{2n+1}) + d(v, a_{2n+1})}] \\ &\leq sd(v, a_{2n+2}) + sa_1d(v, a_{2n+1}) + sa_2[\frac{d(v, Sv)d(a_{2n+1}, a_{2n+2})}{d(v, a_{2n+2}) + d(a_{2n+1}, Sv) + d(v, a_{2n+1})}] \\ &\quad + sa_3[\frac{d(v, Tv)d(a_{2n+1}, a_{2n+2})[1 + d(a_{2n+1}, Sv)d(v, a_{2n+1})]}{d(v, a_{2n+2}) + d(v, a_{2n+1})}]\end{aligned}$$

4

which implies that

$$\begin{aligned}|z| &= |d(v,Sv)| \\ &\leq s|d(v,a_{2n+2})| + sa_1|d(v,a_{2n+1})| \\ &+ sa_2[\frac{|d(v,Sv)||d(a_{2n+1},a_{2n+2})|}{|d(v,a_{2n+2})| + |d(a_{2n+1},Sv)| + |d(v,a_{2n+1})|}] \\ &+ sa_3[\frac{|d(v,Tv)|d(a_{2n+1},a_{2n+2})|[1+|d(a_{2n+1},Sv)||d(v,a_{2n+1})|]}{|d(v,a_{2n+2})| + |d(v,a_{2n+1})|}]\end{aligned} \tag{4}$$

Taking the limit of as $n \to \infty$, we obtain that $|z| = |d(v,Sv)| \leq 0$, a contradiction with .so $|z| = 0$.Hence $Sv = v$.Similarly,we obtain $Tv = v$.

Uniqueness: To prove this,assume that $v^* \neq v$ is another common fixed point of S and T.Then

$$\begin{aligned}d(v,v^*) &= d(Sv,Tv^*) \\ &\leq a_1 d(v,v^*) + a_2[\frac{d(v,Sv)d(v^*,Tv^*)}{d(v,Tv^*) + d(v^*,Sv) + d(v,v^*)}] \\ &+ a_3[\frac{d(v,Tv)d(v^*,Tv^*)[1+d(v^*,Sv)d(v,v^*)]}{d(v,Tv^*) + d(v,v^*)}]\end{aligned}$$

So that

$$\begin{aligned}|d(v,v^*)| &= |d(Sv,Tv^*)| \\ &\leq a_1|d(v,v^*)| + a_2[\frac{|d(v,Sv)||d(v^*,Tv^*)|}{|d(v,Tv^*)| + |d(v^*,Sv)| + |d(v,v^*)|}] \\ &+ a_3[\frac{|d(v,Tv)||d(v^*,Tv^*)|[1+|d(v^*,Sv)||d(v,v^*)|]}{|d(v,Tv^*)| + |d(v,v^*)|}] \\ &\leq a_1|d(v,v^*)|\end{aligned}$$

which is a contradiction.So that $v = v^*$ which proves the uniqueness of common fixed point.
Now, we consider the second case:
Step 1 : $d(a,Tb) + d(b,Sa) + d(a,b) = 0$.Take $a = a_{2n}$ and $b = a_{2n+1}$ in this expression, we get, $d(a_{2n},Ta_{2n+1}) + d(a_{2n+1},Sa_{2n}) + d(a_{2n},a_{2n+1}) = 0$
$\Longrightarrow d(Sa_{2n},Ta_{2n+1}) = 0$
So that ,$a_{2n} = Sa_{2n} = a_{2n+1} = Ta_{2n+1} = a_{2n+2}$
Thus,we have $a_{2n+1} = Sa_{2n} = a_{2n}$, So there exist H_1 and R_1 such that
$H_1 = SR_1 = R_1$ where $H_1 = a_{2n+1}$ and $R_1 = a_{2n}$.
Using foregoing arguments,one can also show that there exist H_2 and R_2 such that$H_2 = TR_2 = R_2$ where $H_2 = a_{2n+2}$ and $R_2 = a_{2n+1}$.
As, $d(R_1,TR_2) + d(R_2,SR_1) + d(R_1,R_2) = 0$ (by definition) we have,
$d(SR_1,TR_2) = 0$,hence $H_1 = SR_1 = TR_2 = H_2$.
Therefore we obtain $H_1 = SR_1 = SH_1$.
Similarly,we have $H_2 = TH_2$.
As $H_1 = H_2 \Rightarrow SH_1 = TH_1 = H_1$,hence $H_1 = H_2$ is common fixed point of S and T.
For uniqueness of common fixed point ,assume that H_1^* in X is another common fixed point of S and T.Then we have $SH_1^* = TH_1^* = H_1^*$
As $d(H_1,TH_1^*) + d(H_1^*,SH_1) + d(H_1,H_1^*) = 0$,
hence $d(H_1,H_1^*) = d(SH_1,TH_1^*) = 0$
This implies that $H_1 = H_1^*$.

Step 2 : $d(a,Tb) + d(a,b) = 0$.Take $a = a_{2n}$ and $b = a_{2n+1}$ in this expression, we get, $d(a_{2n},Ta_{2n+1}) + d(a_{2n},a_{2n+1}) = 0$

5

$\Longrightarrow d(Sa_{2n}, Ta_{2n+1}) = 0$
So that ,$a_{2n} = Sa_{2n} = a_{2n+1} = Ta_{2n+1} = a_{2n+2}$
Thus,we have $a_{2n+1} = Sa_{2n} = a_{2n}$, So there exist H_3 and R_3 such that
$H_3 = SR_3 = R_3$ where $H_3 = a_{2n+1}$ and $R_3 = a_{2n}$.
Using foregoing arguments, one can also show that there exist H_4 and R_4 such that$H_4 = TR_4 = R_4$ where $H_4 = a_{2n+2}$ and $R_4 = a_{2n+1}$.
As, $d(R_3, TR_4) + d(R_3, R_4) = 0$ (by definition) we have,
$d(SR_3, TR_4) = 0$, hence $H_3 = SR_3 = TR_4 = H_4$.
Therefore we obtain $H_3 = SR_3 = SH_3$.
Similarly,we have $H_4 = TH_4$.
As $H_3 = H_4 \Rightarrow SH_3 = TH_3 = H_3$,
Hence $H_3 = H_4$ is common fixed point of S and T.
For uniqueness of common fixed point ,assume that H_3^* in X is another common fixed point of S and T. Then we have $SH_3^* = TH_3^* = H_3^*$
As $d(H_3, TH_3^*) + d(H_3, H_3^*) = 0$,
hence $d(H_3, H_3^*) = d(SH_3, TH_3^*) = 0$
This implies that $H_3 = H_3^*$.
This completes the proof of the theorem. □

Corollary 3.2. [4] Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mappings satisfying

$$\begin{aligned} d(Sa, Tb) \leq a_1 d(a, b) + a_2[\frac{d(a, Sa)d(b, Tb)}{d(a, Tb) + d(b, Sa) + d(a, b)}] \\ + a_3[\frac{d(a, Ta)d(b, Tb)[1 + d(b, Sa)d(a, b)]}{d(a, Tb) + d(a, b)}] \end{aligned}$$

for all $a, b \in X$ such that $a \neq b, d(a, Tb) + d(b, Sa) + d(a, b) \neq 0, d(a, Tb) + d(a, b) \neq 0$ where a_1, a_2, a_3 are nonnegative reals with $sa_1 + s(a_2 + a_3) \prec 1$ or $d(Ta, Tb) = 0$ if $d(a, Tb) + d(b, Ta) + d(a, b) = 0, d(a, Tb) + d(a, b) = 0$. Then T has a unique fixed point in X.

Proof : We can prove this result by applying Theorem **??** with the condition $a_3 = 0$. □

Corollary 3.3. [4] Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mappings satisfying (for some fixed n)

$$\begin{aligned} d(T^n a, T^n b) \leq a_1 d(a, b) + a_2[\frac{d(a, T^n a)d(b, T^n b)}{d(a, T^n b) + d(b, T^n a) + d(a, b)}] \\ + a_3[\frac{d(a, T^n a)d(b, T^n b)[1 + d(b, T^n a)d(a, b)]}{d(a, T^n b) + d(a, b)}] \end{aligned}$$

for all $a, b \in X$ such that $a \neq b, d(a, T^n b) + d(b, T^n a) + d(a, b) \neq 0, d(a, T^n b) + d(a, b) \neq 0$ where a_1, a_2, a_3 are nonnegative reals with $sa_1 + s(a_2 + a_3) \prec 1$ or $d(Ta, Tb) = 0$ if $d(a, T^n b) + d(b, T^n a) + d(a, b) = 0, d(a, T^n b) + d(a, b) = 0$. Then T has a unique fixed point in X.

Proof : We can prove this result by applying corollary 3.2 with the condition $T = T^n$. From corollary 3.2, we obtain that $v \in X$ such that $T^n v = v$. The uniqueness follows from

$$\begin{aligned} d(Tv, v) &= d(TT^n v, T^n v) \\ &= d(T^n Tv, T^n v) \\ &\leq a_1 d(Tv, v) + a_2[\frac{d(Tv, T^n Tv)d(v, T^n v)}{d(Tv, T^n v) + d(v, T^n Tv) + d(Tv, v)}] \\ &\quad + a_3[\frac{d(Tv, T^n Tv)d(v, T^n v)[1 + d(v, T^n Tv)d(Tv, v)]}{d(Tv, T^n v) + d(Tv, v)}] \\ d(Tv, v) &\leq a_1 d(Tv, v) \end{aligned} \tag{5}$$

By taking modulus of 5 and since $a_1 < 1$,we obtain
$|d(Tv, v)| \leq a_1 |d(Tv, v)| < |d(Tv, v)|$,a contradiction.
So $Tv = v$. Therefore $Tv = T^n v = v$.Hence,the fixed point of T is unique.
This completes the proof. □

Theorem 3.4. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be self-mappings from X into itself satisfying*

$$\begin{aligned} d(Sa, Tb) \leq a_1 d(a, b) + a_2[\frac{d^2(a, Tb) + d^2(b, Sa)}{d(a, Tb) + d(b, Sa)}] + a_3[d(a, Sa) + d(b, Tb)] \\ + a_4[d(a, b) + d(b, Sa)] + a_5[\frac{d^2(b, Tb)}{d(a, Tb) + d(a, b)}] \end{aligned} \tag{6}$$

for all $a, b \in X$ such that $a \neq b, d(a, Tb) + d(b, Sa) \neq 0, d(a, Tb) + d(a, b) \neq 0$ where a_1, a_2, a_3, a_4 and a_5 are nonnegative reals with $a_1 + 2(sa_2 + a_3) + a_4 + a_5 \prec 1$ or $d(Sa, Tb) = 0$ if $d(a, Tb) + d(b, Sa) = 0, d(a, Tb) + d(a, b) = 0$. Then S and T have a unique common fixed point.

Proof : Let $a_0 \in X$, define sequence $\{a_n\}$ in X such that

$$\begin{aligned} a_{2n+1} &= Sa_{2n}, \\ a_{2n+2} &= Ta_{2n+1} \forall n \geq 0. \end{aligned}$$

Now, we show that the sequence $\{a_n\}$ is Cauchy sequence.
Let $a = a_{2n}$ and $b = a_{2n+1}$ in 6; we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(Sa_{2n}, Ta_{2n+1}) \\ &\leq a_1 d(a_{2n}, a_{2n+1}) + a_2[\frac{d^2(a_{2n}, Ta_{2n+1}) + d^2(a_{2n+1}, Sa_{2n})}{d(a_{2n}, Ta_{2n+1}) + d(a_{an+1}, Sa_{2n})}] \\ &\quad + a_3[d(a_{2n}, Sa_{2n}) + d(a_{2n+1}, Ta_{2n+1})] \\ &\quad + a_4[d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, Sa_{2n})] \\ &\quad + a_5[\frac{d^2(a_{2n+1}, Ta_{2n+1})}{d(a_{2n}, Ta_{2n+1}) + d(a_{2n}, a_{2n+1})}] \\ &\leq a_1 d(a_{2n}, a_{2n+1}) + a_2[\frac{d^2(a_{2n}, a_{2n+2}) + d^2(a_{an+1}, a_{2n+1})}{d(a_{2n}, a_{2n+2}) + d(a_{an+1}, a_{2n+1})}] \\ &\quad + a_3[d(a_{2n}, a_{2n+1}) + d(a_{an+1}, a_{2n+2})] \\ &\quad + a_4[d(a_{2n}, a_{2n+1}) + d(a_{an+1}, a_{2n+1})] \\ &\quad + a_5[\frac{d^2(a_{2n+1}, a_{2n+2})}{d(a_{2n}, a_{2n+2}) + d(a_{2n}, a_{2n+1})}] \end{aligned}$$

So that,

$$\begin{aligned}|d(a_{2n+1},a_{2n+2})| &\leq a_1|d(a_{2n},a_{2n+1})| + a_2[\frac{|d^2(a_{2n},a_{2n+1})|}{|d(a_{2n},a_{2n+2})|}] \\ &\quad + a_3[|d(a_{2n},a_{2n+1})| + |d(a_{an+1},a_{2n+2})] + a_4|d(a_{2n},a_{2n+1})| \\ &\quad + a_5|d(a_{2n+1}+a_{2n+2})| \\ &\leq a_1|d(a_{2n},a_{2n+1})| + sa_2[|d(a_{2n},a_{2n+1}) + |d(a_{2n+1},a_{2n+2})|] \\ &\quad + a_3[|d(a_{2n},a_{2n+1})| + |d(a_{an+1},a_{2n+2})] + a_4|d(a_{2n},a_{2n+1})| \\ &\quad + a_5|d(a_{2n+1}+a_{2n+2})| \\ &\leq (a_1+sa_2+a_3+a_4)|d(a_{2n},a_{2n+1})| + (sa_2+a_3+a_5)|d(a_{2n+1},a_{2n+2})|\end{aligned}$$

$d(a_{2n+1},a_{2n+2}) \leq \dfrac{(a_1+sa_2+a_3+a_4)}{1-(sa_2+a_3+a_5)}|d(a_{2n},a_{2n+1})|$
$d(a_{2n+1},a_{2n+2}) \leq \lambda|d(a_{2n},a_{2n+1})|$
$where \lambda = \dfrac{(a_1+sa_2+a_3+a_4)}{1-(sa_2+a_3+a_5)} < 1$
Similarly we obtain,
$d(a_{2n+2},a_{2n+3}) \leq \lambda|d(a_{2n+1},a_{2n+2})|$
Therefore, with $\lambda = \dfrac{(a_1+sa_2+a_3+a_4)}{1-(sa_2+a_3+a_5)} < 1$ and for all $n \geq 0$ and consequently, we have

$$\begin{aligned}|d(a_{2n+1},a_{2n+2})| &\leq \lambda|d(a_{2n},a_{2n+1})| \\ &\leq \lambda^2|d(a_{2n-1},a_{2n})| \leq \cdots \\ &\leq \lambda^{2n+1}|d(a_0,a_1)|\end{aligned}$$

That is, for all $n \in \mathbb{N}$, we can write

$$\begin{aligned}|d(a_{n+1},a_{n+2})| \leq \lambda|d(a_n,a_{n+1})| \leq \lambda^2|d(a_{n-1},a_n)| \\ \leq \cdots \leq \lambda^{n+1}|d(a_0,a_1)|\end{aligned} \quad (7)$$

Thus for any $m > n; m,n \in \mathbb{N}$, we get

$$\begin{aligned}|d(a_n,a_m)| &\leq s|d(a_n,a_{n+1})| + s|d(a_{n+1},a_m)| \\ &\leq s|d(a_n,a_{n+1})| + s^2|d(a_{n+1},a_{n+2})| + s^2|d(a_{n+2},a_m)| \\ &\leq \cdots \\ &\leq s|d(a_n,a_{n+1})| + s^2|d(a_{n+1},a_{n+2})| + s^3|d(a_{n+2},a_{n+3})| \\ &\quad + \cdots + s^{m-n-1}|d(a_{m-2},a_{m-1})| + s^{m-n}|d(a_{m-1},a_m)|.\end{aligned}$$

By using 7, we get

$$\begin{aligned}|d(a_n,a_m)| &\leq s\lambda^n|d(a_0,a_1)| + s^2\lambda^{n+1}|d(a_0,a_1)| + s^3\lambda^{n+2}|d(a_0,a_1)| \\ &\leq \cdots \leq +s^{m-n-1}\lambda^{m-2}|d(a_0,a_1)| + s^{m-n}\lambda^{m-1}|d(a_0,a_1)| \\ &= [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}]|d(a_0,a_1)| \\ &= s\lambda^n[1 + s\lambda + s^2\lambda^2 + \cdots + s^{m-n-1}\lambda^{m-n-1}]|d(a_0,a_1)| \\ &= s\lambda^n\sum_{k=0}^{\infty}(s\lambda)^k|(d(a_0,a_1))| \\ &\leq \frac{s\lambda^n}{1-s\lambda}|d(a_0,a_1)|\end{aligned}$$

and hence, $|d(a_n, a_m)| \leq \frac{s\lambda^n}{1-s\lambda}|d(a_0, a_1)| \to 0$ as $n \to \infty$
Thus, $\{a_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $v \in X$ such that $a_n \to v$ as $n \to \infty$. Assume not, then there exists $z \in X$ such that

$$|d(v, Sv)| = |z| > 0 \tag{8}$$

So by using the triangular inequality and 6, we get

$$\begin{aligned}
z &= d(v, Sv) \\
&\leq sd(v, a_{2n+2}) + sd(a_{2n+2}, Sv) \\
&= sd(v, a_{2n+2}) + sd(Ta_{2n+1}, Sv) \\
&\leq sd(v, a_{2n+2}) + sa_1 d(v, a_{2n+1}) + sa_2[\frac{d^2(v, Ta_{2n+1}) + d^2(a_{2n+1}, Sv)}{d(v, Ta_{2n+1}) + d(a_{2n+1}, Sv)}] \\
&\quad + sa_3[d(v, Sv) + d(a_{2n+1}, Ta_{2n+1})] \\
&\quad + sa_4[d(v, a_{2n+1}) + d(a_{2n+1}, Sv)] + sa_5[\frac{d^2(a_{2n+1}, Ta_{2n+1})}{d(v, Ta_{2n+1}) + d(v, a_{2n+1})}] \\
&= sd(v, a_{2n+2}) + sa_1 d(v, a_{2n+1}) + sa_2[\frac{d^2(v, a_{2n+2}) + d^2(a_{2n+1}, Sv)}{d(v, a_{2n+2}) + d(a_{2n+1}, Sv)}] \\
&\quad + sa_3[d(v, Sv) + d(a_{2n+1}, a_{2n+2})] \\
&\quad + sa_4[d(v, a_{2n+1}) + d(a_{2n+1}, Sv)] + sa_5[\frac{d^2(a_{2n+1}, a_{2n+2})}{d(v, a_{2n+2}) + d(v, a_{2n+1})}]
\end{aligned}$$

which implies that

$$\begin{aligned}
|z| &= |d(v, Sv)| \\
&\leq s|d(v, a_{2n+2})| + sa_1|d(v, a_{2n+1})| + sa_2[\frac{|d^2(v, a_{2n+2})| + |d^2(a_{2n+1}, Sv)|}{|d(v, a_{2n+2})| + |d(a_{2n+1}, Sv)|}] \\
&\quad + sa_3[|d(v, Sv)| + |d(a_{2n+1}, a_{2n+2})|] \\
&\quad + sa_4[|d(v, a_{2n+1})| + |d(a_{2n+1}, Sv)|] + sa_5[\frac{|d^2(a_{2n+1}, a_{2n+2})|}{|d(a_{2n+1}, a_{2n+2)}|}] \\
&\leq (sa_2 + sa_3 + sa_4)|d(v, Sv)| \\
|z| &\leq |z|
\end{aligned} \tag{9}$$

Taking the limit of 9 as $n \to \infty$, we get a contradiction with 8. so $|z| = 0$. Hence $Sv = v$. Similarly, we obtain $Tv = v$.
Uniqueness: To prove this, let us assume that $v^* \neq v$ is another common fixed point of S and T. Then

$$\begin{aligned}
d(v, v^*) &= d(Sv, Tv^*) \\
&\leq a_1 d(v, v^*) + a_2[\frac{d^2(v, Tv^*) + d^2(v^*, Sv)}{d(v, Tv^*) + d(v^*, Sv)}] \\
&\quad + a_3[d(v, Sv) + d(v^*, Tv^*) \\
&\quad + a_4[d(v, v^*) + d(v^*, Sv)] + a_5[\frac{d^2(v^*, Tv^*)}{d(v, Tv^*) + d(v, v^*)}] \\
d(v, v^*) &\leq (a_1 + 2a_2 + 2a_4)d(v, v^*)
\end{aligned}$$

which is a contradiction. So that $v = v^*$ which proves the uniqueness of common fixed point in X.

For the second case:
$d(Sa,Tb)=0$ if $d(a,Tb)+d(b,Sa)=0$, the proof of unique common fixed point can be completed in the line of Theorem 3.4.
This completes the proof of the theorem. □

Corollary 3.5. [4] Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mappings satisfying

$$d(Sa,Tb) \leq a_1 d(a,b) + a_2[\frac{d^2(a,Tb)+d^2(b,Sa)}{d(a,Tb)+d(b,Sa)}] + a_3[d(a,Sa)+d(b,Tb)]$$
$$+ a_4[d(a,b)+d(b,Sa)] + a_5[\frac{d(^2b,Tb)}{d(a,Tb)+d(a,b)}]$$

for all $a,b \in X$ such that $a \neq b, d(a,Tb)+d(b,Sa) \neq 0, d(a,Tb)+d(a,b) \neq 0$ where a_1,a_2,a_3,a_4 and a_5 are nonnegative reals with $a_1+2(sa_2+a_3)+a_4+a_5 \prec 1$ or $d(Sa,Tb)=0$ if $d(a,Tb)+d(b,Sa)=0, d(a,Tb)+d(a,b)=0$. Then T have a unique common fixed point.

Proof : We can prove this result by applying Theorem 3.4 with the condition $a_4 = a_5 = 0$. □

Corollary 3.6. [4] Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mappings satisfying (for some fixed n)

$$d(T^n a, T^n b) \leq a_1 d(a,b) + a_2[\frac{d^2(a,T^n b)+d^2(b,T^n a)}{d(a,T^n b)+d(b,T^n a)}] + a_3[d(a,T^n a)+d(b,T^n b)]$$
$$+ a_4[d(a,b)+d(b,T^n a)] + a_5[\frac{d^2(b,T^n b)}{d(a,T^n b)+d(a,b)}]$$

for all $a,b \in X$ such that $a \neq b, d(a,T^n b)+d(b,T^n a) \neq 0, d(a,T^n b)+d(a,b) \neq 0$ where a_1,a_2,a_3,a_4 and a_5 are nonnegative reals with $a_1+2(sa_2+a_3)+a_4+a_5 < 1$ or $d(T^n a,T^n b)=0$ if $d(a,T^n b)+d(b,T^n a)=0, d(a,T^n b)+d(a,b)=0$. Then T have a unique common fixed point in X.

Proof : We can prove this result by applying corollary 3.5 with the condition $T = T^n$. From corollary 3.5, we obtain that $v \in X$ such that $T^n v = v$. The uniqueness follows from

$$\begin{aligned} d(Tv,v) &= d(TT^n v, T^n v) \\ &= d(T^n Tv, T^n v) \\ &\leq a_1 d(Tv,v) + a_2[\frac{d^2(Tv,T^n v)+d(v,T^n Tv)}{d(Tv,T^n v)+d(v,T^n Tv)}] \\ &\quad + a_3[d(Tv,T^n Tv)+d(v,T^n v) \\ &\quad + a_4[d(Tv,v)+d(v,T^n Tv)] + a_5\frac{d(v,T^n v)}{(Tv,T^n v)+d(Tv,v)} \\ d(Tv,v) &\leq (a_1+2a_2+2a_4)d(Tv,v) \end{aligned} \tag{10}$$

By taking modulus of 10 and since $(a_1+2a_2+2a_4) < 1$,we obtain
$|d(Tv,v)| \leq (a_1+2a_2+2a_4)|d(Tv,v)| < |d(Tv,v)|$,a contradiction.
So $Tv = v$.Therefore $Tv = T^n v = v$. Hence,the fixed point of T is unique.
This completes the proof. □

4 Acknowledgements

The authors thank the editor and referees for their useful comments and suggestions.

References

[1] Banach, S., Sur les operations dans les ensembles abstraits et leur application aux equations integrales,Fundam.Math.3,(1922),133-181.

[2] Bakhtin, IA, "The contraction principle in quasi metric spaces", *In: Functional Analysis*, vol. 30, pp. 26-37 (1989).

[3] Rao, K.P.R, Swamy, P. R and Prasad, J. R, "A common fixed point theorem in Complex Valued b-metric spaces", *Bulletin of Mathematics and Statistics Research*, vol. 1, no. 1, pp. 1-8, 2013.

[4] A. K. Dubey, Complex valued-metric Spaces and common fixed point theorems under rational contractions. Journal of Complex Analysis, 2016.

[5] Czerwik, "Nonlinear set-valued contractions mappings in b-metric spaces", Atti del Seminario Matematico e Fisico dell' Universita di Modena, Vol. 46, n0. 2, pp. 263-276, 1998.

[6] Mukheimer. A. A, "Some common fixed point theorems in complex valued b-metric spaces", The scientific world Journal, Vol. 2014,Article ID 587825, 6 pages, 2014.

[7] Azam. A, Fisher. B, and Khan. M, " Common fixed point theorems in complex valued metric Spaces", Numerical Functional Analysis and Optimization, Vol. 32, no. 3, pp. 243-253, 2011.