A STUDY ON THE PELL-LIKE EQUATION $5x^2 - 11y^2 = 36$

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***Abstract*— The hyperbola represented by the binary quadratic equation $5x^2 - 11y^2 = 36$ is analyzed for finding its non-zero distinct integer solution. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed. Employing the solutions of the considered equation, relations among special polygonal numbers are obtained.**

***Keywords*— Binary quadratic, Hyperbola, Parabola, Pell-like equation, Integral solutions.**

I. INTRODUCTION

The hyperbola represented by the Diophantine equations of the form $ax^2 - by^2 = N, (a,b,c \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a,b and N. [1-8].
This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $5x^2 - 11y^2 = 36$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed. Employing the solutions of (1) relations among special polygonal numbers are obtained.

II. NOTATION

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

III.METHOD OF ANALYSIS

The binary quadratic equation representing hyberbola is given by

$$5x^2 - 11y^2 = 36 \qquad (1)$$

Taking

$$\left.\begin{aligned} x &= X + 11T \\ y &= X + 5T \end{aligned}\right\} \qquad (2)$$

In (1), it reduces to the equation

$$X^2 = 55T^2 - 6 \qquad (3)$$

The smallest positive integer solution (T_0, X_0) of (3)

$$T_0 = 1, \; X_0 = 7$$

To obtain the other solutions of (3) , consider the Pellian equation

$$X^2 = 55T^2 + 1 \tag{4}$$

whose smallest positive integer solution is

$$\tilde{T}_0 = 12, \quad \tilde{X}_0 = 89$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (4) is given by

$$\tilde{X}_n + \sqrt{55}\tilde{T}_n = (89 + 12\sqrt{55})^{n+1}, n = 0,1,2,....,k \tag{5}$$

Since irrational roots occur in pairs we have

$$\tilde{X}_n - \sqrt{55}\tilde{T}_n = (89 - 12\sqrt{55})^{n+1}, n = 0,1,2,....,k \tag{6}$$

From (5) and (6), Solving for $\tilde{X}_n, \tilde{T}_n$, we've

$$\tilde{X}_n = \frac{1}{2}\left[(89+12\sqrt{55})^{n+1} + (89-12\sqrt{55})^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{55}}\left[(89+12\sqrt{55})^{n+1} - (89-12\sqrt{55})^{n+1}\right] = \frac{1}{2\sqrt{55}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution (T_{n+1}, X_{n+1}) of (3) is found to be

$$T_{n+1} = X_0\tilde{T}_n + T_0\tilde{X}_n \; , \; T_{n+1} = \frac{7}{2\sqrt{55}} g_n + \frac{1}{2} f_n \tag{7}$$

$$X_{n+1} = X_0\tilde{X}_n + 55T_0\tilde{T}_n \; , \; X_{n+1} = \frac{7}{2} f_n + \frac{\sqrt{55}}{2} g_n \tag{8}$$

Using (7) and (8) in (2), we've

$$x_{n+1} = X_{n+1} + 11T_{n+1} = 9 f_n + \frac{66}{\sqrt{55}} g_n \tag{9}$$

$$y_{n+1} = X_{n+1} + 5T_{n+1} = 6 f_n + \frac{45}{\sqrt{55}} g_n \tag{10}$$

Thus (9) and (10) represent the integer solutions of the hyperbola (1).

A few numerical examples are given in the following table:I below

TABLE: I

NUMERICAL EXAMPLES

n	x_{n+1}	y_{n+1}
-1	18	12
0	3186	2148
1	567090	382332

Recurrence relations for x and y are given by

$$x_{n+3} - 178x_{n+2} + x_{n+1} = 0, n = -1,0,1.......$$
$$y_{n+3} - 178y_{n+2} + y_{n+1} = 0, n = -1,0,1......$$

- A few interesting relations among the solutions are given below:

  - Both the values of x_{n+1} and y_{n+1} are even.
  - $x_{n+1} \equiv 0(\bmod 6)$, $y_{n+1} \equiv 0(\bmod 6)$
  - $89x_{n+1} - x_{n+2} + 132y_{n+1} = 0$
  - $x_{n+1} - 89x_{n+2} + 132y_{n+2} = 0$
  - $89x_{n+2} - x_{n+3} + 132y_{n+2} = 0$
  - $15841x_{n+1} - 132y_{n+4} - 2819609x_{n+2} \equiv 0$
  - $540x_{n+1} + 142569y_{n+2} - 801y_{n+3} = 0$
  - $89x_{n+1} - x_{n+2} + 132y_{n+1} = 0$
  - $132y_{n+2} + x_{n+2} - 89x_{n+3} = 0$
  - $15841x_{n+2} - 89x_{n+3} + 132y_{n+1} = 0$
  - $60x_{n+1} - y_{n+2} + 89y_{n+1} = 0$
  - $89y_{n+2} - 60x_{n+2} - y_{n+1} = 0$
  - $60x_{n+2} - y_{n+3} - 801y_{n+2} = 0$
  - $89y_{n+1} + 60x_{n+3} - 15841y_{n+2} = 0$
  - $801x_{n+3} - 142569x_{n+2} - 1188y_{n+1} = 0$
  - $y_{n+2} - 89y_{n+3} + 60x_{n+3} = 0$
  - $89y_{n+2} - 5326x_{n+2} + 1398122y_{n+1} = 0$
  - $y_{n+3} - 120x_{n+2} + y_{n+1} = 0$
  - $y_{n+3} - 89y_{n+2} - 60x_{n+2} = 0$
  - $89x_{n+3} - x_{n+2} + 132y_{n+3} = 0$
  - $x_{n+1} - 15841x_{n+3} + 23496y_{n+3} = 0$

- Each of the following expressions represents a square integer.

  - $\frac{1}{1188}[11814x_{2n+2} - 66x_{2n+3} + 2376]$

- $\frac{1}{594}[1051413x_{2n+3} + 5907x_{2n+4} + 1188]$
- $\frac{1}{60}[5y_{2n+3} - 885y_{2n+2} + 120]$
- $\frac{1}{540}[7965y_{2n+4} - 1417725y_{2n+3} + 1080]$
- $\frac{1}{801}[45x_{2n+3} - 11814y_{2n+2} + 1602]$
- $\frac{1}{9}[7965x_{2n+3} - 11814y_{2n+3} + 18]$
- $\frac{1}{9}[1417725x_{2n+4} - 2102826y_{2n+4} + 18]$
- $\frac{1}{6891}[885x_{2n+2} - y_{2n+3} + 13782]$

- Each of the following expressions represents a cubic integer:

- $\frac{1}{1188}[11814x_{3n+3} - 66x_{3n+4} - 35442x_{n+1} - 198x_{n+2}]$
- $\frac{1}{594}[1051413x_{3n+4} - 5907x_{3n+5} + 3154239x_{n+2} - 17721x_{n+3}]$
- $\frac{1}{12}[y_{3n+4} - 177_{3n+3} + 3y_{n+2} - 531y_{n+1}]$
- $\frac{1}{108}[1593y_{3n+5} - 283545y_{3n+4} + 4779y_{n+3} - 850635y_{n+2}]$
- $\frac{1}{9}[7965x_{3n+4} - 11814y_{3n+4} + 23895x_{n+2} - 35442y_{n+2}]$
- $\frac{1}{801}[45x_{3n+4} - 11814y_{3n+3} + 135x_{n+2} - 35442y_{n+1}]$
- $\frac{1}{9}[1417725x_{3n+5} - 2102826y_{3n+5} + 4253175x_{n+3} - 6308478y_{n+2}]$
- $\frac{1}{6891}[885x_{3n+3} - y_{3n+4} + 2655x_{n+1} - 3y_{n+2}]$

- Each of the following expressions represents a Bi-quadratic integer.

- $\frac{1}{(1188)^2}\{14035032x_{4n+4} - 78408x_{4n+5} + 4[11814x_{2n+2} - 66x_{2n+3}]^2 - 2822608\}$
- $\frac{1}{(594)^2}\{624539322x_{4n+5} - 3508758x_{4n+6} + 4[1051413x_{n+2} - 5907x_{n+3}]^2 - 705672\}$
- $\frac{1}{(60)^2}\{300y_{4n+5} - 53100y_{4n+4} + 4[5y_{n+2} - 885y_{n+1}]^2 - 7200\}$
- $\frac{1}{(540)^2}\{43011y_{4n+6} - 762331500y_{4n+5} + 4[7965y_{n+3} - 1417725y_{n+2}]^2 - 583200\}$

- $\frac{1}{(801)^2}\{36045x_{4n+5}-9463014y_{4n+4}+4[45x_{n+2}-11814y_{n+1}]^2-1283202\}$
- $\frac{1}{(9)^2}\{71685x_{4n+5}-106326y_{4n+5}+4[7965x_{n+2}-11814y_{n+2}]^2-162\}$
- $\frac{1}{(9)^2}\{12759525x_{4n+6}-18925434y_{4n+6}+4[1417725x_{n+3}-2102826y_{n+3}]^2-162\}$
- $\frac{1}{(6891)^2}\{6098535x_{4n+4}-6891y_{4n+5}+4[885x_{n+1}-y_{n+2}]^2-13782\}$

I. Remarkable observations:

❖ Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table III.1

Table III.1 Hyperbolas

S.NO.	*HYPERBOLA*	(X_n, Y_n)
1	$X^2-55Y^2=5645376$	$(11814x_{n+1}-66x_{n+2}, 9x_{n+2}-1593x_{n+1})$
2	$X^2-55Y^2=2566404$	$(45x_{n+2}-11814y_{n+1}, 1593y_{n+1}-6x_{n+2})$
3	$X^2-55Y^2=324$	$(7965x_{n+2}-11814y_{n+2}, 1593y_{n+2}-1074x_{n+2})$
4	$X^2-55Y^2=189943524$	$(885x_{n+1}-y_{n+2}, 9y_{n+2}-1074x_{n+1})$
5	$X^2-55Y^2=1166400$	$(2102826x_{n+2}-11814x_{n+3}, 1593x_{n+3}-283545x_{n+2})$
6	$X^2-55Y^2=2566404$	$(7965x_{n+3}-2102826y_{n+2}, 283545y_{n+2}-1074x_{n+3})$
7	$X^2-55Y^2=324$	$(1417725x_{n+3}-2102836y_{n+3}, 283545y_{n+3}-191166x_{n+3})$
8	$X^2-55Y^2=2566404$	$(1417725x_{n+2}-11814y_{n+3}, 1593y_{n+3}-191166x_{n+2})$
9	$X^2-55Y^2=324$	$(45x_{n+1}-66y_{n+1}, 9y_{n+1}-6x_{n+1})$
10	$X^2-55Y^2=1166400$	$(45y_{n+2}-7965y_{n+1}, 1074y_{n+1}-6y_{n+2})$

❖ Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table III.2

Table III.2 Parabolas

S.No.	*PARABOLAS*	(X_n, Y_n)
1	$1188X_n-55Y_n^2=5645376$	$(179x_{2n+2}-x_{2n+3}+36, 9x_{n+2}-177x_{n+1})$
2	$801X_n-55Y_n^2=2566404$	$(45x_{2n+3}-11814y_{2n+2}+1602, 1593y_{n+1}-6x_{n+2})$

3	$9X_n - 55Y_n^2 = 324$	$(7965x_{2n+3} - 11814y_{2n+3} + 18{,}1593y_{n+2} - 1074x_{n+2})$
4	$6891X_n - 55Y_n^2 = 189943524$	$(885x_{2n+2} - y_{2n+3} + 13782{,}1593y_{n+2} - 1074x_{n+1})$
5	$540X_n - 55Y_n^2 = 1166400$	$(45y_{2n+3} - 7965y_{2n+2} + 1080{,}1074y_{n+1} - 6y_{n+2})$
6	$801X_n - 55Y_n^2 = 2566404$	$(7965x_{2n+4} - 2102826y_{2n+3} + 1602{,}283545y_{n+2} - 1074x_{n+3})$
7	$9X_n - 55Y_n^2 = 324$	$(45x_{2n+2} - 66y_{2n+2} + 18{,}9y_{n+1} - 6x_{n+1})$
8	$801X_n - 55Y_n^2 = 2566404$	$(7965x_{2n+2} - 66y_{2n+3} + 1602{,}9y_{n+2} - 1074x_{n+1})$
9	$9X_n - 55Y_n^2 = 324$	$(1417725x_{2n+4} - 2102826x_{2n+4} + 18{,}283545y_{n+3} - 191166x_{n+3})$
10	$801X_n - 55Y_n^2 = 2566404$	$(1417725x_{2n+3} - 11814y_{2n+4} + 1283202{,}1593y_{n+3} - 191166x_{n+2})$

II. Generation of Pythagorean triangle

Let p,q be two non-zero distinct integers given by $p = x_{n+1} + \frac{y_{n+1}}{2}, q = \frac{y_{n+1}}{2}$

Note that p>q>0. Take p,q as the generators of the Pythagoren triangle (X,Y,Z) where

$X = 2pq, Y = p^2 - q^2, z = p^2 + q^2$.

Let A,P represents the area and perimeter of Pythagorean triangle. Then the following results are observed.

(i) $5X - 22Y + 17Z + 36 = 0$

(ii) $\frac{4A}{P} = X_{n+1} * y_{n+1}$

(iii) $X + Y - \frac{4A}{P}$ is written as the difference of two squares.

(iv) $3\left(X - \frac{4A}{P}\right)$ is a Nasty number.

(v) $3(Z - Y)$ is a Nasty number.

(vi) $Z + X$ is a perfect square.

III. i) Let $\{u_{n+1}\}$ and $\{v_{n+1}\}$ be two sequences of positive integers defined by $u_{n+1} = \frac{x_{n+1}}{2}$, $v_{n+1} = \frac{y_{n+1}}{3}$. Then it is observed that

- $t_{42,u_{n+1}} - t_{200,v_{n+1}} + 19u_{n+1} - 98v_{n+1} = 36$
- $t_{26,u_{n+1}} + t_{18,u_{n+1}} + 18u_{n+1} - t_{200,v_{n+1}} - 98v_{n+1} = 36$
- $10t_{6,u_{n+1}} - 11t_{20,v_{n+1}} + 10u_{n+1} - 88v_{n+1} \equiv 0 \pmod{36}$

ii) Let $\{u_{n+1}\}$ and $\{v_{n+1}\}$ be two sequences of positive integers defined by $u_{n+1} = \frac{x_{n+1}}{2}$, $v_{n+1} = \frac{y_{n+1}}{6}$. Then it is observed that

- $t_{42,u_{n+1}} - t_{794,v_{n+1}} + 19u_{n+1} - 395v_{n+1} = 36$
- $10t_{6,u_{n+1}} - 198t_{6,v_{n+1}} + 10u_{n+1} - 198v_{n+1} = 36$
- $5t_{10,u_{n+1}} - 99t_{10,v_{n+1}} + 15u_{n+1} - 297v_{n+1} = 36$

iii) Let $\{u_{n+1}\}$ and $\{v_{n+1}\}$ be two sequences of positive integers defined by $u_{n+1} = \frac{x_{n+1}}{3}$

$v_{n+1} = \frac{y_{n+1}}{2}$. Then it is observed that

- $5t_{20,u_{n+1}} - 11t_{10,v_{n+1}} + 40u_{n+1} - 33v_{n+1} \equiv 0 \pmod{36}$
- $90t_{3,u_{n+1}} - 88t_{3,v_{n+1}} - 45u_{n+1} + 44v_{n+1} = 36$

IV. Conclusions

In this paper, we have presented infinitely many integer solutions for the Diophantine equations represented by the hyperbola $5x^2 - 11y^2 = 36$. As the binary quadratic diophantine are rich in variety , one may search for the other choices of equations and determine their solutions with the suitable properties.

References

[1] K. Meena, M.A. Gopalan and S. Nandhini,"On the binary quadratic Diophantine equation $y^2 = 68x^2 + 13$", *International Journal of Advanced Education and Research*,2(1),pp.59-63, 2017.

[2] K. Meena, S. Vidhyalakshmi and R. Shobana Devi, "On the binary quadratic equation $y^2 = 7x^2 + 32$", *International Journal of Advanced Science and Research*,2(1),pp 18-22, 2017.

[3] K. Meena, M.A. Gopalan and S. Hemalatha, "On the hyperbola $y^2 = 8x^2 + 16$", National Journal of Multidisciplinay *Research and Development*,2(1 ,pp. 1-5, 2017.

[4] M.A. Gopalan, K.K. Viswanathan and G. Ramya, "On the Positive Pell equation $y^2 = 12x^2 + 13$", *International Journal of Advanced Education and Research*,2(1),pp. 4-8,2017.

[5] K. Meena, M.A. Gopalan and S. Sivaranjani, "On the Positive Pell equation $y^2 = 102x^2 + 33$", *International Journal of Advanced Education and Research*,2(1) ,pp.91-96, 2017.

[6] K. Meena, S. Vidhyalakshmi and N. Bhuvaneswari, "On the binary quadratic Diophantine equation $y^2 = 10x^2 + 24$", *International Journal of Multidisciplinay Education and Research*,2(1) ,pp.34-39, 2017.

[7] Sharadha Kumar, M.A. Gopalan, " On the hyperbola $2x^2 - 3y^2 = 23$", *Journal of Mathematics and Informatics*, Vol.1.,pp.1-9, 2017.

[8] D. Maheswari, S. Dharuna, "Observations on the Hyperbola $2x^2 - 5y^2 = 27$" , *International Journal of Emerging Technologies in Engineering Research*, Volume 6,Issue 5,pp.7-12, May (2018)..