COMPLEX IDEMPOTENT IN C[G], AND THE PROBLEM OF FUTURE RESEARCH

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ABSTRACT

This research paper is based on the concept of the complex idempotents in C[G]. Here C[G] is the group ring of G over the complex numbers C. To find the idea of the complex idempotents in C[G] we have used the concept of Hermitian inner product on C[G], norms in C[G] as well as the absolute value of the elements of C[G]. We have taken idempotents e in the complex group rings C[G] and proved that $tr\, e \geq 0$. When group G is finite and K is a field of char K = 0 then $0 \leq tr\, e \leq 1$ as $0 \leq \dim Ve \leq |G|$, and also *tr e* is always in the prime subfield of K. There is a basic difference in these two properties of *tr e*.

(i) When *tr e* is contained in the prime subfield of K then it is an algebraic property.

(ii) When the inequality $0 \leq tr\, e \leq 1$, then it is an analytic property.

Thus we have obtained the analytic assertion $0 \leq tr\, e \leq 1$ and the algebraic assertion on the values of *tr e*. Such concept, when char of field K is p>0 is still a problem, of research in future.

Keywords: Complex idempotent, Hermitian inner product, Norms in C[G], Algebraic property, Analytic property of *tr e*.

1. Introduction: This paper presents an idea of complex idempotents in C[G]. For a finite group G and group ring K[G] we choose a homomorphism *f* such that,

$f : K[G] \to M_n(K)$, here $M_n(K)$ is a matrix of dimension *n* and *n* = *dim V* = |*G*|. *f(x)* is a permutation matrix of 0s and 1s having one 1 in each row and column. If $x \neq 1$ then for all $x \in G,\ tr\, f(x) = 0$, but when *x* = 1 then *f*(*x*) is an identity matrix and $tr\, f(1) = n = |G|$.

Let us suppose that $\alpha = \sum a_x . x \in K[G],$ then $tr\, f(\alpha) = \sum a_x . tr\, f(x)$

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$= a_{1.}|G|$. Similarly, for an arbitrary G, we define a map, $tr\,K[G]\rightarrow K$ and so, $tr\left(\sum a_x.x\right)=a_1$. Now we take some lemmas for basic results.

Lemma 1: Let $tr:K[G]\rightarrow K$ be *K*-linear then for all $\alpha,\beta\in K[G]$ we have $tr\,\alpha\beta=tr\beta\alpha$.

Lemma 2: Let us suppose that G be a finite group and char K Θ |G|, then

(i) If $\alpha\in K[G]$ is nilopotent $\Rightarrow tr\,\alpha=0$

(ii) If $e\in K[G]$ is an idempotent $\Rightarrow tr\,e=(\dim K[G].e)/|G|$

2. Complex group ring C[G].

Let us suppose that G be an arbitrary group and C[G] be the group ring of G over the complex numbers C. Let α,β are elements in C[G], and $\alpha=\sum a_x.x,\ \beta=\sum b_x.x,$ then the inner product [1][2] and norms [1] in C[G] as follow,

$$(\alpha,\beta)=\sum a_x\bar{b}_x$$

$$\|\alpha\|=(\alpha,\alpha)^{1/2}=\left(\sum|a_x|^2\right)^{1/2}$$

3. Let *e* be an idempotent in C[G] and we have to prove $tr\,e\geq 0$.

Lemma 3: The Hermitian inner product [5] on C[G] will be, if $\alpha,\beta\in C[G]$,

$(\alpha,\beta)=tr\,\alpha\,\bar{\beta}=tr\,\bar{\beta}\alpha$. If a map $\alpha\rightarrow\bar{\alpha}$ is a ring antiautomorphism [3] of C[G] for all α,β,γ we have

$$(\alpha,\beta\gamma)=(\alpha\bar{\gamma},\beta)=(\bar{\beta}\alpha,\gamma)$$

4. Decomposition of C[G] as direct sum of two right ideals.

Let $I=eC[G]$ be the right ideal of C[G] generated by the idempotent *e* and let I$^{\cap}$ be its orthogonal complement. Since G is finite so C[G] is a finite dimensional vector space. Hence I + I$^{\cap}$ = C[G] is a direct sum decomposition. Here I$^{\cap}$ is also a right ideal of C[G]. If $\alpha\in I$ $\beta\in I^{\perp}$ and $\gamma\in C[G]$ then $\alpha\bar{\gamma}\in I$ (right ideal of C[G]). Hence

$(\alpha,\beta\gamma)=(\alpha\bar{\gamma},\beta)=0$ and $\beta\gamma$ is orthogonal to all $\alpha\in I$, also we get that $\beta\gamma\in I^{\perp}$. Therefore, we have found that $I+I^{\perp}=C[G]$ is a decomposition of C[G] as a direct sum of two right ideals.

Now, we suppose that,

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$g + g^{\cap} = 1$, is similar decomposition of 1. Thus g and $g^{\cap}$ are idempotents with, $I = gC[G]$ and $I^{\perp} = g^{\perp}C[G]$. As g is orthogonal to $g^{\perp}C[G]$ so $g^{\perp}C[G] = (1-g)C[G]$ and for all $\alpha \in C[G]$ we have $(g,(1-g)\alpha) = (\overline{(1-g)}g, \alpha) = 0$ also $\overline{(1-g)}g \in C[G]^{\perp} = 0 \Rightarrow g = \bar{g}g$. Again $\bar{g} = \overline{\bar{g}g} = g\bar{g} = g$ and hence g is a self-adjoint [4] idempotent. Since, e C[G] = I = g C[G], as e, g are left identities of ideal I, hence, we have

$tr\ e = tr\ ge = tr\ eg = tr\ g$. But

$g = \bar{g}\,g$ so, $tr\ e = tr\ g = tr\ g\,\bar{g} = \|g\|^2 \geq 0$

Therefore, $tr\,e \geq 0$.

5. We have obtained that $tr\,e \geq 0$ but still it remained to show that $tr\,e \in Q$, here Q is the set of rational numbers.

Since we get that I + $I^{\cap}$ = C[G] for finite G, but $tr\,e \in Q$ is possible for infinite group G. Such decompositions are not true for infinite dimensional inner product spaces. So there are two ways to get this problem's solution.

(i) First way was used by **Kaplansky (69) [7]** and **Montgomery (69)[8]**

Both of them embedded C[G] in some larger algebra in which these decompositions are possible. These larger algebras were defined on suitable topologies on C[G].

(ii) Second approach was given by **Passman (71)** [6]. His observations were based on element g, as taken above. Let us take $\alpha \in I$ and distance between α and $1 \in$ C[G] as follow

$d(\alpha,1)^2 = \|\alpha - 1\|^2 = (\alpha - 1, \alpha - 1)$

Since $g + g^{\cap} = 1$ and $(\alpha - g, g^{\cap}) = 0$

we have

$d(\alpha,1)^2 = \|\alpha - g\|^2 + \|g^{\cap}\|^2$. So $d(\alpha,1) \geq \|g^{\cap}\|$, and becomes equal if only if $\alpha = g$. Thus g is unique element of I and closest to 1. Now let G be an arbitrary group and L is a complex subspace of C[G] then, $d(L, \gamma) = \inf_{\alpha \in L} \|\alpha - \gamma\|$. Hence, there exists a sequence of elements of I whose corresponding distances approach to d (I,1). This sequence plays the role of the element g.

Lemma 4: If $\alpha, \beta \in L$, a linear subspace of C[G], then

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$$|(\beta, \alpha-\gamma)|^2 \leq \|\beta\|^2 \left(\|\alpha-\gamma\|^2 - d(L,\gamma)^2\right)$$

Lemma 5: If $\alpha, \beta \in C[G]$, then

(i) $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$, $|\alpha+\beta| \leq |\alpha| + |\beta|$

(ii) $|tr\ \alpha| \leq \|\alpha\|$, $(\alpha, 1) = tr\ \alpha$

(iii) $\|\alpha\beta\| \leq \|\alpha\|.|\beta|$, $|\alpha\beta| \leq |\alpha|.|\beta|$

6. When idempotent of C[G] be *e*!0

Now we suppose that $e \neq 0$ be an idempotent in C[G] and take I = *e* C[G]. Then I is a linear subspace [9] of C[G]. We take *d* = *d*(I,1) be the distance between I and 1. For each integer *n*>0 we get $f_n \in I$ with

$$\|g_n - 1\|^2 < d^2 + 1/n^4$$

Lemma 6: If there exist non-negative real constants *r'* and *r"* then

(i) $\left| \|g_n\|^2 - tr\, g_n \right| \leq r'/n$

(ii) $\|g_n.e - e\| \leq r''/n$

Lemma 7: If the trace of *e* is real then $tr\, e \geq \|e\|^2 / |e|^2 > 0$

Proof - From lemmas 5(ii) and 6 we get

$$\left| \|g_n\|^2 - tr\, g_n \right| < r'/n$$

$| tr\ g_n\ e - tr\ e| < r''/n$ and $tr\ g_n.\ e = tr\ e.\ g_n = tr\ g_n$. Since $g_n \in e$ C[G] $\Rightarrow e\ g_n = g_n$. So we have from the above

$\left| \|g_n\|^2 - tr\, e \right| \leq (r'+r'')/n$ and $tr\ e = \lim_{n\to\infty} \|g_n\|^2$. Thus *tr e* is real and non-negative.

Now from lemma 5 (i) (iii) and 6 (ii) we get

$\|e\| \leq \|e - g_n e\| + \|g_n.e\| \leq r''/n + \|g_n\|.\ |e|$.

By taking limit as $n \to \infty$ we obtain

$$\|e\| \leq (tr\ e)^{\frac{1}{2}}.|e| \text{ and}$$

hence,

$$tr\, e \geq \frac{\|e\|^2}{|e|^2} > 0$$

7. An analytic proof of *tr e* is algebraic over *Q*.

Theorem 8 [Kaplansky (69)][7] – Let K be a field of characteristic 0 and let $e \neq 0,1$ be an idempotent in K[G]. Then *tr e* is a totally real algebraic number with the property that it and its algebraic conjugates lie strictly between 0 and 1.

Proof - Let us take $e = \sum b_x.x \in K[G]$ be an idempotent and F is finitely generated field extension of the rational number Q. It is given by $F = Q(b_x \mid x \in Supp e)$. Thus $e \in F[G] \subseteq K[G]$. Now F is embedded in the complex numbers C so that

$e \in F[G] \subseteq C[G]$, and we can suppose *e* as an element of C[G]. Here *e* is an idempotent with same trace and $e \neq 0$, 1. Now by lemma 7, we have *tr e* > 0. As 1 – *e* is also a non – zero idempotent of C[G], so *tr* (1 - *e*) = 1 – *tr e* > 0 and *tr e* < 1.

Now we suppose σ as any field automorphism of the complex numbers. Then σ includes a ring automorphism of C[G] by

$$\because \alpha = \sum a_x.x \Rightarrow \alpha^\sigma = \sum a_x^\sigma.x$$

Since e^σ is again an idempotent of C[G], and $tr\, e^\sigma = (tr\, e)^\sigma$. Therefore we get $0 < (tr\, e)^\sigma < 1$ for all such σ. But if *tr e* were transcendental [10] over *Q*, then there would exist a field automorphism σ with $(tr\ e)^\sigma$ not real. It is a contradiction. Therefore, *tr e* is algebraic over *Q*.

Hence, the other two idempotents of K[G] are 0 and 1 also their traces are 0 and 1.

8. von Neumann finite Ring –

A ring R is said as von Neumann finite [11] if $\alpha.\beta = 1$ in R$\Rightarrow \beta\alpha = 1$.

Corollary 9: If K is a field of characteristic 0, then K[G] is von Neumann finite.

Proof - Let us take that $\alpha, \beta \in K[G]$ and $\alpha\beta = 1$. We take $e = \beta\alpha$. Then, $e^2 = \beta\alpha.\beta\alpha = \beta(\alpha\beta)\alpha = \beta\alpha = e.$ Therefore *e* is an idempotent in K[G]. But from lemma 1, *tr e*= *tr* $\beta\alpha$ = *tr* $\alpha\beta$=1. Hence by theorem 8, we have *e* =1.

9 What are to be investigated yet in complex idempotent?

These all conditions are problems of research, when the characteristic of field is greater than 0, in future.

(i) If K is a field of characteristic $p > 0$, them K[G] is von Neumann finite.

(ii) In a group ring C[G], *tr e* ≥ 0

(iii) *tr e* is algebraic in *Q*.

(iv) Analytic assertion $0 \le tr\, e \le 1$.

10. Conclusions: In this research paper, we have presented the concept of complex idempotent in C[G] in easy way. For this purpose we have used Hermitian inner product as well as norm. With the help of these concept we get result as *tr e* ≥ 0. We have got that *tr e* is algebraic over *Q*. It is not transcendental over *Q*. Therefore, we obtained, the analytic assertion $0 \le tr\, e \le 1$ and the algebraic assertion on the values of *tr e*. This has been found when characteristic of field *K* is 0. But, when the characteristic of field *K* is $p > 0$ then such concept of complex idempotent of C[G] is not known. It is still a matter of research in future.

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