ON SUMS AND PRODUCTS OF $k-J$ -STAR DAGGER MATRICES

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Abstract:

In this paper, the concept of $k-J$ -star dagger matrices are introduced. Necessary and sufficient conditions are determined for a sums and products of $k-J$ -star dagger matrices are $k-J$ -star dagger. Also $k-J$ -star dagger concept is applied on bi-EP, bi-normal matrices. For which some equivalent conditions are given.

Keywords:

EP matrix, k -hermitian, J -hermitian, and permutation matrix, indefinite inner product space, star dagger matrix, moore penrose, group inverse.

AMS Classification:

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Introduction:

Through out this paper, we shall deal with $C_{n\times n}$ the space of $n\times n$ complex matrices. Let C_n be the space of complex n-tuples. A matrix A is called EP_r if $\rho(A)=r$ and $N(A)=N(A^*)$ (or) $R(A)=R(A^*)$ where $\rho(A)$ and denotes the rank of A. $N(A)$ and $R(A)$ denote the null space and range space of A respectively. Let k -be a fixed product of disjoint transpositions is $S_n=\{1,2,\cdots n\}$ Since k is involutary, it can be verified that the associated permutation matrix K satisfying the following condition $K=K^T=K^{-1}$ and $k(x)=Kx, K^2=I$. A matrix $A=(a_{ij})\in C_{n\times n}$ is k -hermitian if $a_{ij}=a_{k(j)k(i)}$ for $i,j=1,2\cdots n$. A theory for k -hermitian matrices is developed in [8]. For, $x=(x_1,x_2\cdots x_n\}^T\in C_n$. Let us define the function $k(x)=(x_{k(1)},x_{k(2)}\cdots x_{k(n)}\}^T\in C_n$.

Moreover, the indefinite matrix product in C_n is a sesquilinear form $[x,y], x,y\in C^n$ defined by the equation $[x,y]=<x,Jy>$. Here $(.,.)$ is the standard Euclidean inner product, J is an invertible hermitian matrix. We make an additional assumption that $J^2=I$. Let $J_n\in C_{n\times n}$ be such that $J_n=J_n^*=J_n^{-1}$. The indefinite matrix product of matrices $A\in C_{n\times n}$ and $B\in C_{n\times n}$ is defined by $A\circ B=AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ where $A^{[*]}=J_nA^*J_m$, [6,7] where J_m and J_n weights in appropriate spaces. A matrix $A\in C_{n\times n}$ is said to be $k-J$ - EP if it satisfies the condition equivalently $N(A)=N(KA^{[*]}K)=N(KJA^*JK)$.

For a pair matrices A and B, $[A,B]=AB-BA$ is the commutator A of and B. For A, let $A^{\dagger}$ and A^* denote the Moore-Penrose inverse [2] and the conjugate transpose of A respectively. A matrix A is called star-dagger if star of

A commutes with the dagger of A, that is $[A^*, A^\dagger] = O.$ The class of star-dagger matrices is denoted by SD. This class includes the class of partial isometries. A matrix A is called a partial isometry [3] if $A^\dagger = A^*$.

A matrix A is called a group matrix if there is a solution to the equations: $AXA = A,\ XAX = X$ and $AX = XA$. This solution, if it exists is unique and is called the group inverse $A^\#$ of A [5]. A matrix A is called EP if $R(A) = R(A^*)$ (or) equivalently if $[A^\dagger, A] = O$, Where $R(A)$ denotes the range space of A [9,10]. We shall denote these classes of matrices by GP and EP respectively [4]. A matrix A is called bi-normal, bi-EP or bi-dagger if $[A^*A, AA^*] = O$, $[A^\dagger A, AA^\dagger] = O$ (or) $(A^\dagger)^2 = (A^2)^\dagger$ respectively. In [3], the inter relationship between these classes are studied in detail. The purpose of this paper is to explore some additional properties of SD matrices.

$k-J$ -STAR DAGGER AND OTHER CLASS OF MATRICES

Lemma:1

Let A be both $k-J$ -Star dagger and bi- $k-J-EP$. Then (i) $[AKA^{[*]}K, A^\dagger A] = 0$ and (ii) $[KA^{[*]}KA, AA^\dagger] = 0$.

Proof:

Let $A \in k-J-SD \bigcap$ {bi- $k-J-EP$ }. Then we have $A^\dagger KA^{[*]}K = KA^{[*]}KA^\dagger$

$A^\dagger KJA^*JK = KJA^*JKA^\dagger$ (1)

And $AA^\dagger A^\dagger A = A^\dagger AAA^\dagger$ (2)

Now $AKA^{[*]}KA^\dagger A = AA^\dagger KA^{[*]}KA$

$AKJA^*JKA^\dagger A = AA^\dagger KJA^*JKA = AA^\dagger A^\dagger AKJA^*JKA$ (by 1)

$= A^\dagger AAA^\dagger KJA^*JKA$ (by 2)

This together with equation (1) imply that $AKJA^*JKA^\dagger A = A^\dagger AAKJA^*JKA^\dagger A$

Hence we have $AKJA^*JKA^\dagger A$ is $k-J-$ hermitian and hence $[AA^{[*]}, A^\dagger A] = O$ that is $[AKJA^*JK, A^\dagger A] = O.$

Again, by equation (1), we have $[AKJA^*JK, A^\dagger A] = O.$ (3)

By (i), $AKJA^*JKA^\dagger A$ is $k-J-$ hermitian and hence equation (3) implies that $AA^\dagger KJA^*JKA$ is $k-J-$ hermitian and therefore $[KJA^*JKA, AA^\dagger] = O.$

Theorem 2:

Let $A \in C_{n\times n}$ be a $k-J-$ star dagger and bi- $k-J-EP$ matrix. Then A is bi- $k-J-$ normal.

Proof:

Consider $AKA^{[*]}KKA^{[*]}KA = A(KA^{[*]}KAA^\dagger)(A^\dagger AKA^{[*]}K)A$

$AKJA^*JKKJA^*JKA = A(KJA^*JKAA^\dagger)(A^\dagger AKJA^*JK)A = AKJA^*JK(AA^\dagger A^\dagger A)KJA^*JKA = AKJA^*JK(A^\dagger AAA^\dagger)KJA^*JKA$

$= A(KJA^*JKA^\dagger)AA(A^\dagger KJA^*JK)A \quad = (AA^\dagger KJA^*JKA)AKJA^*JKA^\dagger A \quad = (KJA^*JKAAA^\dagger)AKJA^*JKA^\dagger A \quad = KJA^*JKAAKJA^*JKA^\dagger A$
$= KJA^*JKA(AKJA^*JKA^\dagger A) \quad = KJA^*JKA(A^\dagger AAKJA^*JK) = KJA^*JK(AA^\dagger A)AKJA^*JK$. That is,
$AKJA^*JKKJA^*JKA = KJA^*JKAAKJA^*JK$, this implies that $[KJA^*JKA, AKJA^*JK] = O.$

Remark:3

For any two square matrices of order n, we have the following

$[KJA^*JKA, BKJB^*JK] = O \Leftrightarrow KJA^*JKABKJB^*JK$ is $k-J-$ hermitian $\Rightarrow KJA^*JKABKJB^*JK$ is $k-J-EP$

$\Leftrightarrow (AB)^\dagger = B^\dagger A^\dagger \quad \Rightarrow [AA^\dagger, BB^\dagger] = O.$

Theorem:4

Let $A, B \in C_{n\times n}$ be such that $KJB^*JKA^\dagger = B^\dagger KJA^*JK$. Then the following are equivalent:

(i) $[KJA^*JKA, BKJB^*JK] = O$

(ii) $KJA^*JKABKJB^*JK$ is $k-J-$ hermitian

(iii) $KJA^*JKABKJB^*JK$ is $k-J-$ normal

(iv) $KJA^*JKABKJB^*JK$ is $k-J-EP$

(v) $(AB)^\dagger = B^\dagger A^\dagger$

(vi) $[A^\dagger A, BB^\dagger] = O$

Proof:

(i)$\Rightarrow$ (ii)$\Rightarrow$ (iii)$\Rightarrow$ (iv)$\Rightarrow$ (v)$\Rightarrow$ (vi) are obvious.

Let $C = KJA^*JKA$ and $D = BKJB^*JK$ (4)

Then C and D are hpsd and hence $CC^\dagger = C^\dagger C$, $DD^\dagger = D^\dagger D$

Also, $A^\dagger = (KJA^*JKA)^\dagger KJA^*JK$

$A^\dagger A = (KJA^*JKA)^\dagger KJA^*JKA = C^\dagger C = CC^\dagger$. (5)

Similarly, we get $BB^\dagger = DD^\dagger = D^\dagger D.$ (6)

Again, $KJB^*JKA^\dagger = B^\dagger KJA^*JK$ implies that $KJB^*JKA^\dagger = B^\dagger KJA^*JK$ This together with equations (4),(5) and (6) imply that $DCC^\dagger = DD^\dagger C$ (7)

Taking conjugate transpose on both sides of equation (7) we get $CC^\dagger D = CDD^\dagger$ (8)

(vi)$\Rightarrow$ (i)

If (6) holds, then $[A^\dagger A, BB^\dagger] = O$. This means that $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A.$ This together with equations (5) and (6) gives that $C^\dagger CDD^\dagger = DD^\dagger C^\dagger C.$ (9)

Consider, $C^{\dagger}D = C^{\dagger}CDD^{\dagger} = C^{\dagger}(CC^{\dagger}D) = C^{\dagger}CDD^{\dagger} = DD^{\dagger}C^{\dagger}C$ by (8) and (9)

$= DD^{\dagger}CC^{\dagger} = (DD^{\dagger}C)C^{\dagger} = DCC^{\dagger}C^{\dagger}$ by (7)

$= DC^{\dagger}CC^{\dagger} = DC^{\dagger}$.

Thus $C^{\dagger}D = DC^{\dagger}$. Since C is EP, $C^{\dagger}$ is a polynomial in C and hence $CD = DC$. That is $KJA^{*}JKABKJB^{*}JK = BKJB^{*}JKKJA^{*}JKA$. Hence (1) holds.

Example:5

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$, $(AB)^{\dagger} = 1/8\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ and $B^{\dagger}A^{\dagger} = 1/8\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

Therefore, $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Here $rk(A) = rk(B) = rk(AB) = 1$.

But $KB^{[*]}A^{\dagger}K = KJB^{*}JA^{\dagger}K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $KB^{\dagger}A^{[*]}K = KB^{\dagger}JA^{*}JK = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Therefore, $KJB^{*}JA^{\dagger}K = KB^{\dagger}JA^{*}JK$

Example 6:

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $(AB)^{\dagger} = 1/3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $B^{\dagger}A^{\dagger} = 1/9\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Therefore, $(AB)^{\dagger} \neq B^{\dagger}A^{\dagger}$. Here $rk(A) = rk(B) = rk(AB) = 1$.

But $KB^{[*]}A^{\dagger}K = KJB^{*}JA^{\dagger}K = 1/9\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $KB^{\dagger}A^{[*]}K = KB^{\dagger}JA^{*}JK = 1/9\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore, $KJB^{*}JA^{\dagger}K = KB^{\dagger}JA^{*}JK$

Example 7:

Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$AB=\begin{pmatrix}0&0&0\\0&0&1\\0&0&1\end{pmatrix},\ (AB)^{\dagger}=1/2\begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}\ \text{and}\ B^{\dagger}A^{\dagger}=1/2\begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}$$

Therefore, $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$. Here $rk(A)=rk(B)=rk(AB)=1$.

But $KB^{[*]}A^{\dagger}K=KJB^{*}JA^{\dagger}K=1/2\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$ and $KB^{\dagger}A^{[*]}K=KB^{\dagger}JA^{*}JK=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$

Therefore, $KJB^{*}JA^{\dagger}K\neq KB^{\dagger}JA^{*}JK$.

Remark 8:

Let A and B be two partial isometries. Then the product AB is also a partial isometry if and only if $(AB)=B^{\dagger}A^{\dagger}$. If A and B are partial isometries, then we have $B^{\dagger}KA^{[*]}K=KB^{[*]}KA^{\dagger}$. That is $B^{\dagger}KJA^{*}JK=KJB^{*}JKA^{\dagger}$.

Results:9

In [3] Hartwig and Spindelbock have remarked that if A and B are $k-J$ star-orthogonal $k-J-SD$ matrices such that $AB=O=BA$, then $A+B$ will be $k-J-SD$. Here we shall show that $A+B\in k-J-SD$ under some weaker conditions. In the squel we shall make use of some properties of the Moore-Penrose inverse of a matrix [1].

Theorem 10:

Let A and B be $k-J$ -Star orthogonal $k-J-SD$ matrices. Then $A+B\in k-J-SD\in \Leftrightarrow KJA^{*}JKB^{\dagger}+KJB^{*}JKA^{\dagger}=A^{\dagger}KJB^{*}JK+B^{\dagger}KJA^{*}JK$.

Proof:

Since A and B are $k-J$ -Star orthogonal, $(A+B)^{\dagger}=A^{\dagger}+B^{\dagger}$. Now $A+B\in k-J-SD$
$\Leftrightarrow KJ(A+B)^{*}JK(A+B)^{\dagger}=(A+B)^{\dagger}KJ(A+B)^{*}JK \Leftrightarrow KJ(A^{*}+B^{*})JK(A^{\dagger}+B^{\dagger})=(A^{\dagger}+B^{\dagger})KJ(A^{*}+B^{*})JK$

$\Leftrightarrow KJA^{*}JKB^{\dagger}+KJB^{*}JKA^{\dagger}=A^{\dagger}KJB^{*}JK+B^{\dagger}KJA^{*}JK \quad \Leftrightarrow A^{\dagger}KJA^{*}JK+B^{\dagger}KJA^{*}JK+A^{\dagger}KJB^{*}JK+B^{\dagger}KJB^{*}JK$

Remark 11:

Theorem 1 fails if we relax the condition that A and B are $k-J$ -star orthogonal.

Example 12:

Let $A=\begin{pmatrix}1+i&1&0\\1&1+i&0\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}i&0&0\\0&i&0\\0&0&0\end{pmatrix}$, $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ and $J=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$

$A^* = \begin{pmatrix} 1-i & 1 & 0 \\ 1 & 1-i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A^\dagger = 1/6\begin{pmatrix} 1-i & 1 & 0 \\ 1 & 1-i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B^* = \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B^\dagger = 1/2\begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then we get $A, B \in k-J-SD$ and $A+B = \begin{pmatrix} 1+2i & 1 & 0 \\ 1 & 1+2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in k-J-SD$. But A and B are not $k-J$ -star orthogonal. However $AB \neq O \neq BA$. .

$KA^{[*]}KB^\dagger + KB^{[*]}KA^\dagger \neq A^\dagger KB^{[*]}K + B^\dagger KA^{[*]}K$ that is, $KJA^*JKA^\dagger + KJB^*JKA^\dagger \neq A^\dagger KJB^*JK + B^\dagger KJA^*JK$.

Example 13:

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^\dagger = 1/2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B^\dagger = 1/4\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Here $A, B \in k-J-SD$. But A and B are not $k-J$ -star orthogonal matrices such that

$KJA^*JKB^\dagger + KJB^*JKA^\dagger = A^\dagger KJB^*JK + B^\dagger KJA^*JK$ and $A+B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in k-J-SD$. However $AB \neq O \neq BA$.

Remark 14:

Whenever $AB = O = BA$, then by using the formula $A^\dagger = (KJA^*JKA)^\dagger KJA^*JK = KJA^*JK(AKJA^*JK)^\dagger$, $KJB^*JKA^\dagger = O = A^\dagger KJB^*JK$. Similarly using the formula for $B^\dagger$, we have $KJA^*JKB^\dagger = O = B^\dagger KJA^*JK$. Thus $AB = O = BA$ implies $KJA^*JKB^\dagger + KJB^*JKA^\dagger = A^\dagger KJB^*JK + B^\dagger KJA^*JK$. However this condition given in theorem 1 is weaker than that of $AB = O = BA$.

Example 15:

Let $A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^* = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$, $A^\dagger = 1/4\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$, $B^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B^\dagger = 1/4\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Here $A, B \in k-J-SD$, and also A and B are $k-J$ star-orthogonal $k-J-SD$ matrices such that $KJA^*JKB^\dagger + KJB^*JKA^\dagger = A^\dagger KJB^*JK + B^\dagger KJA^*JK$ and $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in k-J-SD$. However $AB \neq O \neq BA$.

Example 16:

Let $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$, $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ and $J=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$, $A^*=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$, $A^\dagger=1/4\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$,

$B^*=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$, $B^\dagger=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$

Here $A,B\in k-J-SD$,, and also A and B are $k-J$ star-orthogonal $k-J-SD$ matrices such that

$KJA^*JKB^\dagger+KJB^*JKA^\dagger=A^\dagger KJB^*JK+B^\dagger KJA^*JK$ and $A+B=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}\in k-J-SD$. However $AB=BA=O$.

Result 17:

Let $A\in k-J-SD$. Then the following are equivalent:

(i) A is bi- $k-J$ - normal

(ii) A is bi- $k-J-EP$.

(iii) A is bi- $k-J$ - dagger.

Proof:

Since $A\in k-J-SD$ the condition $KJA^*JKA^\dagger=A^\dagger KJA^*JK$ automatically holds and the result

follows from theorem (4) by putting $A=B.$

Result 18 :

If A and B are partial isometries, then the following are equivalent:

(i) $[A^\dagger A,BB^\dagger]=O$

(ii) $[KA^{[*]}KA,BKB^{[*]}K]=O$

(iii) $[KA^{[*]}KA,BKB^{[*]}K]=O$

(iv) AB is partial isometry.

Proof:

Since A and B are partial isometries, we have $A^\dagger=KA^{[*]}K$ and $B^\dagger=KB^{[*]}K$ and hence $B^\dagger KA^{[*]}K=KB^{[*]}KA^\dagger$. Therefore by theorem 2, (i),(ii) and (iii) are equivalent.

Theorem 19:

For any two matrices A and B the following are equivalent:

(i) $KJA^{*}JKA = BKJB^{*}JK$

(ii) $A^{\dagger}A = BB^{\dagger}$, $B^{\dagger}KJA^{*}JK = KJB^{*}JKA^{\dagger}$

(iii) $[A^{\dagger}A, BB^{\dagger}] = O$; $R(A) = R(AB)$, $R(KJ(AB)^{*}JK) = R(KJB^{*}JK)$, $B^{\dagger}KJA^{*}JK = KJB^{*}JKA^{\dagger}$

(iv) $[KJA^{*}JKA, BKJB^{*}JK] = O$, $R(A) = R(AB)$, $R(KJ(AB)^{*}JK) = R(KJB^{*}JK)$, $B^{\dagger}KJA^{*}JK = KJB^{*}JKA^{\dagger}$

Proof:

(i) $\Rightarrow$ (ii)

$KJA^{*}JKA = BKJB^{*}JK \Rightarrow R(KJA^{*}JK) = R(B) \Rightarrow A^{\dagger}A = BB^{\dagger}$. Also
$KJA^{*}JKA = BKJB^{*}JK \Rightarrow B^{\dagger}KJA^{*}JKAA^{\dagger} = B^{\dagger}BKJB^{*}JKA^{\dagger} \Rightarrow B^{\dagger}KJA^{*}JK = KJB^{*}JKA^{\dagger}$. Thus (ii) holds.

(ii) $\Rightarrow$ (i)

$B^{\dagger}KJA^{\dagger}JK = KJB^{*}JKA^{\dagger} \Rightarrow BB^{\dagger}KJA^{*}JKA = BKJB^{*}JKA^{\dagger}A$

$\Rightarrow A^{\dagger}AKJA^{*}JKA = BKJB^{*}JKBB^{\dagger} \Rightarrow KJA^{*}JKA = BKJB^{*}JK$. Thus (i) holds. The equivalence if (iii) and (iv) follows from (theorem 2).

Result 20:

For any square matrix A the following are equivalent:

(i) $KJA^{*}JKA = AKJA^{*}JK$

(ii) $A^{\dagger}A = AA^{\dagger}, A^{\dagger}KJA^{*}JK = KJA^{*}JKA^{\dagger}$

(iii) $[A^{\dagger}A, AA^{\dagger}] = O,\ R(A) = R(A^{2}),\ A^{\dagger}KJA^{*}JK = KJA^{*}JKA^{\dagger}$

(iv) $[KJA^{*}JKA, AKJA^{*}JK] = O,\ R(A) = R(A^{2}),\ A^{\dagger}KJA^{*}JK = KJA^{*}JKA^{\dagger}$.

Thus {Normal}= $k-J-SD \cap EP = k-J-SD \cap GP \cap$ {bi- $k-J-EP$ }= $k-J-SD \cap GP \cap$ {bi- $k-J$ -normal}.

Theorem 21:

Let A , $B \in k-J-SD$. If $KJA^{*}JKB^{\dagger} = B^{\dagger}KJA^{*}JK$, $KJB^{*}JKA^{\dagger} = A^{\dagger}KJB^{*}JK$ and then $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Thus $AB \in k-J-SD$.

Proof:

$(AB)^{\dagger}KJ(AB)^{*}JK = B^{\dagger}A^{\dagger}KJ(B^{*}A^{*})JK = B^{\dagger}KJB^{*}JKA^{\dagger}KJA^{*}JK = KJB^{*}JKB^{\dagger}KJA^{*}JKA^{\dagger} = KJB^{*}JKB^{\dagger}KJA^{*}JKA^{\dagger}$

$= KJ(AB)^{*}JK(AB)^{\dagger}$. Thus $AB \in k-J-SD$.

Result 22:

$A \in k-J-SD$. Then $A^2 \in k-J-SD$.

Example 23:

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A^2 = A \in k-J-SD$.

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