CLASSICAL STATISTICAL DISTRIBUTION INVOLVING THE H-FUNCTION OF TWO VARIABLES

Nitesh Kumar Sahu
Department of Mathematics
Govt. Polytechnic College,
Betul (M.P.)

ABSTRACT:

In this paper an attempt has been made to present a unified theory of the classical statistical distributions associated with generalized beta and gamma distributions of one variable. The probability density function is taken in terms of the H-function of two variables. In particular, the characteristic function and the distribution function are investigated.

Keywords: H-function of two variables, Probability Density Function, Characteristic Function, Distribution Function, Classical Statistical Distributions.

1. INTRODUCTION:

In probability theory, several authors have studied a large number of statistical distributions from time to time. For example, Mathai and Saxena [3] introduced a general hypergeometric distribution, whose probability density function involves a hypergeometric functions ${}_2F_1$. Again, Shrivastava and Singhal [5] studied another general class of distributions, whose probability density function involves the H-function. It may be readily seen that the distributions, considered by Mathai and Saxena [3] and all other well-known classical statistical distributions, such as the generalized beta and gamma distributions, the exponential distribution, the generalized F-distribution, students t-distribution, the normal distribution, etc. can be derived as specialized or confluent cases of the class of distributions, considered by Shrivastava and Singhal [5]. More recently, Exton [2] considered the family of distributions which have the probability density function in terms of the product of several generalized hypergeometric functions ${}_pF_q$.

In an attempt to present a further generalization of the probability distributions studied by Shrivastava and Singhal [5], Exton [2] etc. here we introduce and study a general family of statistical probability distributions involving the H-function of two variables.

Recently Mittal and Gupta [4, p. 117] has given the following notation of the H-function of two variables as:

$$H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[{x \atop y}\Big|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)}$$

$$\theta_2(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, ($i = 1, 2, 3$; $j = 2, 3$). Also, all the A's, α's, B's, β's, γ's, δ's, E's, and F's are assumed to the positive quantities for standardization purpose.

The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ ($j = 1, ..., m_2$) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ ($j = 1, ..., n_2$), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ ($j = 1, ..., n_1$) to the left of the contour.

The contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ ($j = 1, ..., m_3$) lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ ($j = 1, ..., n_3$), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ ($j = 1, ..., n_1$) to the left of the contour.

The function, defined by (1), is analytic function of x and y if

$R = \sum_{j=1}^{p_1}\alpha_j + \sum_{j=1}^{p_2}\gamma_j - \sum_{j=1}^{q_1}\beta_j - \sum_{j=1}^{q_2}\delta_j < 0,$

$R = \sum_{j=1}^{p_1}A_j + \sum_{j=1}^{p_3}F_j - \sum_{j=1}^{q_1}B_j - \sum_{j=1}^{q_3}F_j < 0,$

The H-function of two variables given by (1.4.3) is convergent if

$$U = -\sum_{j=n_1+1}^{p_1}\alpha_j - \sum_{j=1}^{q_1}\beta_j + \sum_{j=1}^{m_2}\delta_j - \sum_{j=m_2+1}^{q_2}\delta_j + \sum_{j=1}^{n_2}\gamma_j - \sum_{j=n_2+1}^{p_2}\gamma_j > 0, \qquad (2)$$

$$V = -\sum_{j=n_1+1}^{p_1}A_j - \sum_{j=1}^{q_1}B_j + \sum_{j=1}^{m_3}F_j - \sum_{j=m_3+1}^{q_3}F_j + \sum_{j=1}^{n_3}E_j - \sum_{j=n_3+1}^{p_3}E_j > 0, \qquad (3)$$

and $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$.

2. PROBABILITY DENSITY FUNCTIONS:

This section deals with certain classical statistical distributions associated with beta (or finite) and gamma (or infinite) distributions of one variate. The probability density function is taken in terms of the H-function of two variables. First we find the probability density function.

Let a density function be defined by

$$f(x) = Kx^{\sigma-1}(1-x)^{\rho-1}(1+bx)^{-\sigma-\rho}\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[{\zeta x^{\sigma_1}(1-x)^{\rho_1}(1+bx)^{-\sigma_1-\rho_1} \atop \eta}\middle| {(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right], \quad (4)$$

$0 \le x \le 1$ for finite distribution or generalized beta distribution, and f(x) = 0, elsewhere. If f(x) is a probability density function, then it should satisfy the relation

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} f(x)\,dx \equiv 1 \quad (5)$$

Putting the value of f(x) given by (4) in (5) and evaluating the resulting integral with the help of Mellin-Barnes contour integral for the H-function of two variables given by (1) and the well known definition of beta function (sec, e.g. [1], p.9, Eq. (1)),

$$B(x, y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}dt, \quad (6)$$

Re(x) > 0, Re(y) > 0, we find that

$$K^{-1} = (1+b)^{-\sigma} H^{0,n_1;m_2,n_2+2;m_3,n_3}_{p_1,q_1;p_2+2,q_2+1;p_3,q_3}\left[{\zeta(1+b)^{-\sigma_1} \atop \eta}\middle| {(a_j,\alpha_j;A_j)_{1,p_1}:(1-\sigma,\sigma_1),(1-\rho,\rho_1),(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}(1-\sigma-\rho,\sigma_1+\rho_1):(f_j,F_j)_{1,q_3}}\right], \quad (7)$$

provided Re $(\sigma) + \sigma_1 > 0$, and Re $(\rho) + \rho_1 < 0$.

Again let

$$f(x) = Qe^{-\xi x}x^{\sigma-1} H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[{\zeta x^{\sigma_1} \atop \eta}\middle| {(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right] \quad (8)$$

where $0 < x < \infty$, Re $(\xi) > 0$, Re $(\sigma) > 0$, Re $(\sigma) + \sigma_1 > 0$ and

$$Q^{-1} = \xi^{-\sigma} H^{0,n_1;m_2,n_2+1;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}\left[{\zeta\xi^{-\sigma_1} \atop \eta}\middle| {(a_j,\alpha_j;A_j)_{1,p_1}:(1-\sigma,\sigma_1),(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right] \quad (9)$$

then f(x) will be a probability density function for infinite distribution or generalized gamma distribution. By virtue of (5), we easily get

$$\int_0^{\infty} e^{-\xi x}x^{\sigma-1}\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[{\zeta x^{\sigma_1} \atop \eta}\middle| {(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right] \equiv 1 \quad (10)$$

Now, on evaluating the above integral with the help of (1) and the definition of gamma function (sec, e.g. [1], p.1 Eq (1)),

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{s-1} dt, \quad (11)$$

Re(z) > 0, we obtain the expression for Q given by (9).

3. THE CHARACTERISTIC FUNCTION:

The characteristic function denoted by $\phi(t)$ may be represented as $<e^{itx}>$, where the angle brackets denote "mathematical expectation". We may thus write characteristic function as

$$\phi(t) = <e^{itx}> = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \quad (12)$$

The characteristic function for finite distribution, when f(x) is given by (4), is

$$\phi(t) = K \sum_{r=0}^{\infty} \frac{(it)^r}{r!} H^{0,n_1;m_2,n_2+2;m_3,n_3}_{p_1,q_1;p_2+2,q_2+1;p_3,q_3} \left[{\zeta(1+b)^{-\sigma_1} \atop \eta} \left| {(a_j,\alpha_j;A_j)_{1,p_1}:(1-\sigma-r,\sigma_1),(1-\rho,\rho_1),(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\sigma-\rho-r,\sigma_1+\rho_1):(f_j,F_j)_{1,q_3}} \right. \right], \quad (13)$$

where K is given by (7).

Also, the characteristic function for infinite distribution, when f(x) is given by (8), is

$$\phi(t) = Q(\xi - it)^{-\sigma} H^{0,n_1;m_2,n_2+1;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3} \left[{\zeta(\xi-it)^{-\sigma_1} \atop \eta} \left| {(a_j,\alpha_j;A_j)_{1,p_1}:(1-\sigma,\sigma_1),(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}} \right. \right], \quad (14)$$

where Q is given by (9).

4. CONCLUSION:

Since the H-function of two variables includes almost all the special functions, therefore it can define a very general class of probability model. Thus all the classical statistical distributions mentioned here and elsewhere, will be the special cases of our findings.

REFERECES

1. Erdelyi, A.: Higher Transcendental Functions, Vol.I, McGraw-Hill, New York, 1953.
2. Exton, H.: Handbook of Hypergeometric Integrals, Ellis Horwood Ltd. Chichester, U.K., 1978.
3. Mathi, A. M. & Saxena, R. K.: On a generalized hypergeometric distribution, Metrika 11 (1966), 127-131.

4. Mittal, P. K. and Gupta Gupta, K. C.: An integral involving generalized function of two variables, Proc. Indian Acad. Sci., 75 A, p. 117-123.
5. Shrivastava, H. M. and Singhal, J. P.: On a class of generalized hypergeometric distributions, Jnanabha Sect. A2(1972), 1-9.