INFINITE INTEGRALS INVOLVING H-FUNCTION OF TWO VARIABLES

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ABSTRACT

The aim of this paper is to evaluate some infinite integrals involving the H-function of two variables.

Keywords: H-function of two variable, infinite integrals.

1. INTRODUCTION:

Recently Mittal and Gupta [2, p. 117] has given the following notation of the H-function of two variables as:

$$H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$=\frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)},$$

$$\theta_2(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)}$$

$$\theta_2(\eta)=\frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, (i = 1, 2, 3; j = 2, 3). Also, all the A's, α's, B's, β's, γ's, δ's, E's, and F's are assumed to the positive quantities for standardization purpose.

The contour L_1 is in the ξ-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ (j = 1, ..., m_2) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ (j = 1, ..., n_2), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n_1) to the left of the contour.

The contour L_2 is in the η-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ (j = 1, ..., m_3) lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ (j = 1, ..., n_3), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n_1) to the left of the contour.

The function, defined by (1), is analytic function of x and y if

$R = \sum_{j=1}^{p_1} \alpha_j + \sum_{j=1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j < 0,$

$R = \sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_3} F_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j < 0,$

The H-function of two variables given by (1) is convergent if

$$U = -\sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \gamma_j - \sum_{j=n_2+1}^{p_2} \gamma_j > 0, \tag{2}$$

$$V = -\sum_{j=n_1+1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j > 0, \tag{3}$$

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

In the present investigation we require the following formula:

From Dixon [2]:

$$\int_{-\infty}^{\infty} \frac{\sin(cx)}{\Gamma(\alpha+x)\Gamma(\beta-x)} dx = \frac{[2\cos(\frac{c}{2})]^{\alpha+\beta-2} \sin[\frac{1}{2}c(\beta-\alpha)]}{\Gamma(\alpha+\beta+1)}, \tag{4}$$

provided that Re (α + β) < 1, 0 < c <π.

2. INTEGRALS:

In this section, we shall establish following integrals:

$$\int_{-\infty}^{\infty} \sin(cx)\, H_{p_1,q_1;p_2,q_2+2;p_3,q_3}^{0,n_1;m_2,n_2;m_3,n_3} \left[{\zeta \atop \eta} \Big| {(a_j,\alpha_j;A_j)_{1,p_1} : (c_j,\gamma_j)_{1,p_2} : (e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1} : (d_j,\delta_j)_{1,q_2}, (1-\alpha-x,u), (1-\beta+x,u) : (f_j,F_j)_{1,q_3}} \right] dx$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2} \sin[\tfrac{1}{2}c(\beta-\alpha)]\, H_{p_1,q_1;p_2,q_2+1;p_3,q_3}^{0,n_1;m_2,n_2;m_3,n_3} \left[{(2\cos\frac{c}{2})^{2u}\zeta \atop \eta} \Big| {(a_j,\alpha_j;A_j)_{1,p_1} : (c_j,\gamma_j)_{1,p_2} : (e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1} : (d_j,\delta_j)_{1,q_2}, (-\alpha-\beta,2u) : (f_j,F_j)_{1,q_3}} \right], \tag{5}$$

provided that Re (α + β) < 1, 0 < c <π, |argζ| < ½Uπ, |argη| < ½Vπ, where U and V are given in (2) and (3) respectively.

$$\int_{-\infty}^{\infty} \sin(cx)\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+2,q_2;p_3,q_3}\left[{}^{\zeta}_{\eta}\Big|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\alpha+x,u),(\beta-x,u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2}\sin[\tfrac{1}{2}c(\beta-\alpha)]H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}$$

$$\left[{}^{(2\cos\frac{c}{2})^{-2u}\zeta}_{\eta}\Big|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\alpha+\beta,2u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right], \qquad (6)$$

provided that Re (α + β) < 1, 0 < c <π, |argζ| < ½Uπ, |argη| < ½Vπ, where U and V are given in (2) and (3) respectively.

Proof (5):

The result (5) can be established by replacing the H-function of two variable on the left hand side as contour integral (1), we get

$$\int_{-\infty}^{\infty} \sin(cx)\left[\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\rho,\sigma)\theta_2(\rho)\theta_3(\sigma)\right.$$

$$\left.\frac{1}{\Gamma(\alpha+x+u\rho)\Gamma(\beta-x+u\rho)}\zeta^{\rho}\eta^{\sigma}d\rho d\sigma\right]dx$$

interchanging the order of integral involved in the process, evaluating the inner integral with the help of (4) and applying (1) the definition of H-function of two variables, the value of the integral is obtained. On using the same procedure as above, the integrals (6) are established.

3. PARTICULAR CASES:

On choosing c = π/2 in (5) and (6), we get following results, which are useful in space science and used in explanation of quantum gravitational:

$$\int_{-\infty}^{\infty} \sin(\tfrac{\pi}{2}x)\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+2;p_3,q_3}\left[{}^{\zeta}_{\eta}\Big|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\alpha-x,u),(1-\beta+x,u):(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [\sqrt{2}]^{\alpha+\beta-2}\sin[\tfrac{\pi}{4}(\beta-\alpha)]H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}$$

$$[^{(\sqrt{2})^{2u}\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-\alpha-\beta,2u):(f_j,F_j)_{1,q_3}}], \qquad (7)$$

provided that Re $(\alpha + \beta) < 1$, $|\arg\zeta| < ½ U\pi$, $|\arg\eta| < ½ V\pi$, where U and V are given in (2) and (3) respectively.

$$\int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}x\right) H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+2,q_2;p_3,q_3}$$

$$[^{\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\alpha+x,u),(\beta-x,u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}]dx$$

$$= [\sqrt{2}]^{\alpha+\beta-2} \sin\left[\frac{\pi}{4}(\beta-\alpha)\right] H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}$$

$$[^{(\sqrt{2})^{-2u}\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\alpha+\beta,2u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}], \qquad (8)$$

provided that Re $(\alpha + \beta) < 1$, $\arg\zeta|<½U\pi$, $|\arg\eta| <½V\pi$, where U and V are given in (2) and (3) respectively.

REFERENCES

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