The Forcing Complement Connected Monophonic Number of a Graph

[1]F. Merlin Sugirtha, [2] P.Arul Paul Sudhakar, [3]S.Robinson Chellathurai,
[1]Register Number-12447,
Scott Christian College, Nagercoil-629 003, India.
[2]Assistant Professor, Department of Mathematics,
Rani Anna Government Arts College (W), Tirunelveli-627 012, India.
[3]Associate Professor, Department of Mathematics,
Scott Christian College, Nagercoil-629 003, India.
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012,
Tamil Nadu, India.

Abstract

For a connected graph $G = (V, E)$, let M be a minimum complement connected monophonic set of G. A subset $T \subseteq M$ is called a *forcing subset* for M if M is the unique minimum complement connected monophonic set containing T. A forcing subset for M of minimum cardinality is a *minimum forcing subset of* M. The *forcing complement connected monophonic number* of M, denoted by $f_{m_{cc}}(M)$, is the cardinality of a minimum forcing subset of M. The *forcing complement connected monophonic number* of G, denoted by $f_{m_{cc}}(G)$ is $f_{m_{cc}}(G)$= min$\{f_{m_{cc}}(M)\}$, where the minimum is taken over all minimum complement connected monophonic sets M in G. It is also proved that for every integers a and b with $0 \leq a < b$ and $b > a + 1$, there exists a connected graph G such that $f_{m_{cc}}(G) = a$ and $m_{cc}(G) = b$.

Key Words: monophonic number, complement connected monophonic number, forcing complement connected monophonic number.

AMS Subject Classification: 05C12

1. Introduction:

We consider finite graphs with neither loops nor multiple edges. For any graph G, we denote the vertex set by $V\ (G)$ and the edge set by $E(G)$. We denote the order of by $p = |V(G)|$ and the size by $q = |E(G)|$. The *minimum* and *maximum degrees* of a graph are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively. The subgraph induced by a set of vertices of a graph $G$is denoted by $\langle M \rangle$with $V(\langle M \rangle) = M$ and $E(\langle M \rangle) = \{uv \in E(G): u, v \in M\}$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. For basic graph theoretic terminology we refer to [1]. A *chord* of a path $u_0, u_1, u_2, \ldots, u_n$ is an edge $u_i u_j$ with $j \geq i + 2$. An $u-v$ path is called a *monophonic path* if it is a chordless path. A set M of vertices in a connected graph G is a monophonic set of G if every vertex of G is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of $G$is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* or simply a *$m-$set* of G. A monophonic set M of a connected graph G is said to be a *complement connected monophonic set* if $M = V$ or the subgraph $\langle V - M \rangle$ is connected. The minimum cardinality of a complement connected monophonic set of G is *the complement connected monophonic number* of G and is denoted by $m_{cc}(G)$. The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2. After joining the two graphs the resultant graph will be of diameter atmost 2.

The following theorem is used in sequel.

Theorem:1.1[7]. Every complement connected monophonic set of G contains its extreme vertices.

2.The Forcing Complement Connected Monophonic Number of a Graph

Definition:2.1. Let G be a connected graph and M be a minimum complement connected monophonic set of G. A subset $T \subseteq M$ is called a *forcing subset* for M if M is the unique minimum complement connected monophonic set containing T. A forcing subset for M of minimum cardinality is a *minimum forcing subset of M*. The *forcing complement connected monophonic number* of M, denoted by $f_{m_{cc}}(M)$, is the cardinality of a minimum forcing subset of M. The *forcing complement connected monophonic number* of G, denoted by $f_{m_{cc}}(G)$ is $f_{m_{cc}}(G) = \min\{f_{m_{cc}}(M)\}$, where the minimum is taken over all minimum complement connected monophonic sets M in G.

2

Example:2.2

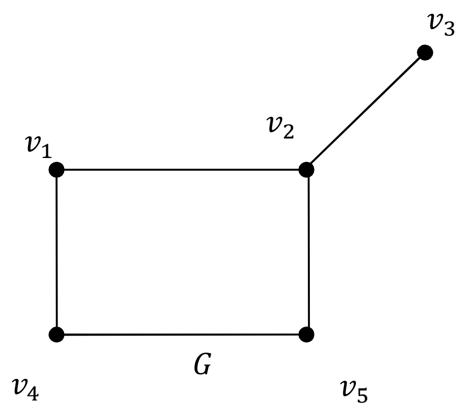

Figure 2.1

For the graph G given in Figure2.1, $M_1 = \{v_1, v_3\}$, $M_2 = \{v_3, v_5\}, M_3 = \{v_3, v_4\}$ are the only three m_{cc}-sets of G such that $f_{m_{cc}}(M_1) = f_{m_{cc}}(M_2) = f_{m_{cc}}(M_3) = 1$ so that $f_{m_{cc}}(G) = 1$.

Theorem: 2.3. For every connected graph G, $0 \leq f_{m_{cc}}(G) \leq m_{cc}(G)$. ∎

Theorem: 2.4. Let G be a connected graph. Then

a) $f_{m_{cc}}(G) = 0$ if and only if G has a unique minimum complement connected monophonic set.

b) $f_{m_{cc}}(G) = 1$ if and only if G has at least two minimum complement connected monophonic sets, one of which is a unique minimum complement connected monophonic set containing one of its elements, and

c) $f_{m_{cc}}(G) = m_{cc}(G)$ if and only if no minimum complement connected monophonic set of G is the unique minimum complement connected monophonic setcontaining any of its proper subsets.

Proof.(a) Let $f_{m_{cc}}(G) = 0$. Then, by definition, $f_{m_{cc}}(M) = 0$ for some minimumcomplement connected monophonic set M of G so that the empty set ϕ is the minimum forcing subset for M. Since the empty set ϕ is a subset of every set, it follows that M is the unique minimumcomplement connected monophonic setof G. The converse is clear.

(b) Let $f_{m_{cc}}(G) = 1$. Then by Theorem 2.4(a), G has at least two minimum complement connected monophonic sets. Also, since $f_{m_{cc}}(G) = 1$, there is a singleton subset T of a minimum complement connected monophonic set M of G such that T is not a subset of any other minimum complement connected monophonic set of G. Thus M is the unique minimum complement connected monophonic set containing one of its elements. The converse is clear.

(c) Let $f_{m_{cc}}(G) = m_{cc}(G)$. Then $f_{m_{cc}}(M) = m_{cc}(G)$ for every minimum complement connected monophonic set M in G. Also, by Theorem 2.3, $m_{cc}(G) \geq 2$ and hence $f_{m_{cc}}(G) \geq 2$. Then by Theorem 2.4(a), G has at least two minimum complement connected monophonic sets and so the empty set ϕ is not a forcing subset for any minimum complement connected monophonic set of G. Since $f_{m_{cc}}(M) = m_{cc}(G)$, no proper subset of M is a forcing subset of M. Thus no minimum complement connected monophonic set of G is the unique minimum complement connected monophonic set containing any of its proper subsets. Conversely, the data implies that G contains more than one minimumcomplement connected monophonic set and no subset of any minimum complement connected monophonic set M other than M is a forcing subset for M. Hence it follows that$f_{m_{cc}}(G) = m_{cc}(G)$. ■

Definition: 2.5. A vertex v of a graph G is said to be a *complement connected monophonic vertex* of G if v belongs to every minimum complement connected monophonic set of G.

Theorem: 2.6. Let G be a connected graph and W be the set of all complement connected monophonic vertices of G. Then $f_{m_{cc}}(G) \leq m_{cc}(G) - |W|$.

Proof. Let M be any minimum complement connected monophonic set of G. Then $m_{cc}(G) = |M|$, $M \subseteq W$ and W is the unique minimum complement connected monophonic set containing $W - M$. Thus $f_{m_{cc}}(G) \leq |W - M| = |W| - |M| = m_{cc}(G) - |M|$. ■

Corollary: 2.7. If G is a connected graph with k extreme vertices, then $f_{m_{cc}}(G) \leq m_{cc}(G) - k$.

Proof. This follows from Theorem 1.1. ■

Theorem: 2.8. For any complete graph $G = K_p (p \geq 2)$ or any non-trivial tree $G = T, f_{m_{cc}}(G) = 0$.

Proof. For $G = K_p$, it follows that the set of all vertices of G is the unique complement connected monophonic set. Hence it follows from Theorem 2.4(a) that $f_m(G) = 0$. For any non-trivial tree G, the complement connected monophonic number $m(G)$ equals the number of end vertices in G. In fact, the set of all end vertices of G is the unique minimum complement connected monophonic set of G and so $f_{m_{cc}}(G) = 0$ by Theorem 2.4(a). ■

Theorem:2.9. For any wheel $G = W_p = K_1 + C_{p-1}$, $f_{m_{cc}}(G) = 2$

Proof. Let $M = \{x, y\}$ be any non adjacent vertices of G. Then M is a complement connected monophonic set of G so that $m_{cc}(G) = 2$. It is clear that no singleton subset of M is a forcing of M and so $f_{m_{cc}}(M) \geq 2$. This is true for all m_{cc}-set M of G so that $f_{m_{cc}}(G) = 2$. ■

Theorem: 2.10. For any cycle $G = C_p (p \geq 5), f_{m_{cc}}(G) = 3$

Proof. Let C_p be $v_1, v_2, \ldots, v_p, v_1$. Let $S_i = \{v_i, v_{i+1}, v_{i+2}\}; 1 \leq i \leq p-2$, $M_1 = \{v_{p-1}, v_p, v_1\}$ and $M_2 = \{v_p, v_1, v_2\}$. Then $S_i (1 \leq i \leq p-2), M_1$ and M_2 are the only m_{cc}-sets of G so that $m_{cc}(G) = 3$.It is easily verified that no singleton or two element subset of any m_{cc}-set of G is a forcing subset of G. Therefore $f_{m_{cc}}(G) = 3$. ■

Theorem: 2.11. For the complete bipartite graph $G = K_{m,n}(1 \leq m \leq n)$,

$$f_{m_{cc}}(G) = \begin{cases} 0, & if\ m = 1, n \geq 2 \\ 3, & if\ m = n = 2 \\ 4, & if\ 3 \leq m \leq n \end{cases}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}, Y = \{y_1, y_2, \ldots, y_n\}$be the bipartite sets of G. When $m = 1$ and $n \geq 2, G$ is a tree with n end vertices. Then by Theorem 2.8, $f_{m_{cc}}(G) = 0$. For $m = 2, n = 2, G$ is the cycle C_4. Then by Theorem 2.10, $f_{m_{cc}}(G) = 3$. For $m \geq 3, n \geq 3, M = \{x_i, x_j, y_r, y_s\}$, where $i \neq j, r \neq s$, is m_{cc}-set of G. It is easily verified that $f_{m_{cc}}(G) \geq 4$. This is true for every m_{cc}-sets M of G. Therefore $f_{m_{cc}}(G) = 4$

Theorem: 2.12. For every integers a and b with $0 \leq a < b$and $b > a + 1$, there exists a connected graph G such that $f_{m_{cc}}(G) = a$ and $m_{cc}(G) = b$.

Proof. Let $P: x, y$ be a path on two vertices and $P_i: u_i v_i; 1 \leq i \leq a$ be a copy of path on two vertices. Let G be a graph obtained from P and $P_i(1 \leq i \leq a)$ by adding new vertices $z_1, z_2, \ldots, z_{b-a-1}$ and joining y with each $z_i(1 \leq i \leq b - a - 1)$. The graph G is shown in Figure 2.2. Let $Z = \{z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of end-vertices of G. First we show that $m_{cc}(G) = b$. Let M be any complement connected monophonic set of G. Then by Theorem 1.1, $Z \subseteq M$. It is clear that Z is not a complement connected monophonic set of G. It is clear that Z is not a complement connected monophonic set of G. Let $H_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. We observe that every m_{cc}-set of G must contain at least one vertex from each H_i $(1 \leq i \leq a)$. Thus $m_{cc}(G) \geq b - a + a = b$. On the other hand since the set$M = Z \cup \{v_1, v_2, v_3, \ldots, v_a\}$ is a complement connected monophonic set of G, it follows that $m(G) \leq |W| = b$. Hence $m_{cc}(G) = b$. Next we show that $f_{m_{cc}}(G) = a$. Every complement connected monophonic set of G contains Z and so it follows from Theorem 2.6 that $f_{m_{cc}}(G) \leq m_{cc}(G) - |Z| = a$. Now, since $m_{cc}(G) = b$ and every m-set of G contains Z, it is easily seen that every m-set M is of the form $Z \cup \{u, u_2, u, \ldots u_a\}$, where $u_i \in H_i(1 \leq i \leq a)$. Let T be any proper subset of M with

$|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \varphi$, which shows that $f_{m_{cc}}(G) = a$. ■

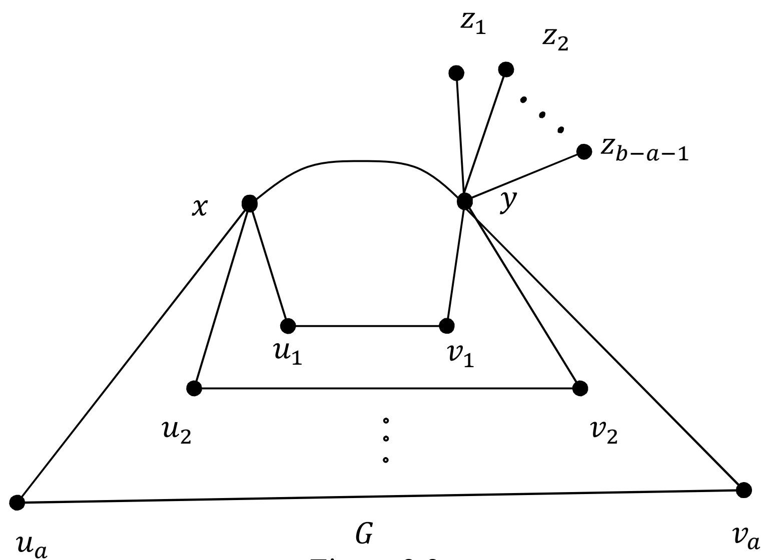

Figure 2.2

3.The Complement connected monophonic and the complement connected forcing monophonic numbers of join of two groups

Theorem 3.1. For the graph $G = K_1 + P_{p-1}$ $(p \geq 4)$, $f_{m_{cc}}(G) = 3$.

Proof. Let x be the vertex of K_1. P_{p-1} be $v_1, v_2, \ldots\ldots, v_{p-1}$. Then $M = \{v_1, v_{p-1}\}$ is a set of all extreme vertices of G. However M is a monophonic set of G. Since $< V - M >$ is connected. M is the unique m_{cc}-set of G so that $m_{cc}(G) = 2$ and $f_{mcc}(G) = 3$. ■

Theorem 3.2. Let $G = \overline{K}_2 + P_{p-2}$ $(p \geq 4)$, Then $m_{cc}(G) = 2$ and $f_{mcc}(G) = 1$.

Proof. Let $V(\overline{K}_2) = \{x, y\}$ and P_{p-2} be $v_1, v_2, \ldots\ldots, v_{p-2}$. Then $M_1 = \{x, y\}$ and $M_2 = \{v_1, v_{p-2}\}$ are the only two m_{cc}-sets of G. Since $< V - M_1 >$ is connected and $< V - M_2 >$ is connected, M_1 and M_2 are the two m_{cc}-set of G such that $f_{mcc}(M_1) = 1$, $f_{mcc}(M_2) = 1$ so that $f_{mcc}(G) = 1$ and $m_{cc}(G) = 2$. ■

7

Theorem 3.3. Let $G = K_{p_1} + K_{p_2}$ $(p_1, p_2 \geq 2)$, Then $m_{cc}(G) = p_1 + p_2$ and $f_{mcc}(G) = 0$.

Proof. Since $G = K_{p_1} + K_{p_2}$ is the complete graph $G = K_{p_1} + K_{p_2}$. $M = V(G)$ is the unique m_{cc}-set of G so that $f_{mcc}(G) = 0$ and $m_{cc}(G) = p_1 + p_2$. ■

Theorem 3.4. Let $G = P_{p_1} + C_{p_2}$ $p_1 \geq 3,\ p_2 \geq 4$. Then $m_{cc}(G) = 2$ and $f_{mcc}(G) = 1$.

Proof. P_{p_1} be $v_1, v_2, \ldots\ldots, v_{p_1}$ and C_{p_2} be $u_1, u_2, \ldots\ldots, u_{p_2}$. Then $M = \{x, y\}$, where x and y are two non adjacent vertices of C_{p_2} and $M_1 = \{v_1, v_{p_1}\}$ are the m_{cc}-sets of G such that $f_{mcc}(M_1) = 1$ and $f_{mcc}(M)$ is either 1 oe 2 so that $f_{mcc}(G) = 1$ and $m_{cc}(G) = 2$.

■

Theorem 3.5. Let $G = P_{p_1} + P_{p_2}$ $(p_1 \geq 3, p_2 \geq 3)$, Then $m_{cc}(G) = 2$ and $f_{mcc}(G) = 1$.

Proof. P_{p_1} be $v_1, v_2, \ldots\ldots, v_{p_1}$ and P_{p_2} be $u_1, u_2, \ldots\ldots, u_{p_2}$. Then $M_1 = \{v_1, v_{p_1}\}$, $M_2 = \{u_1, u_{p_2}\}$ are the only two m_{cc}-sets of G such that $f_{mcc}(M_1) = 1$ and $f_{mcc}(M_2) = 1$ so that $f_{mcc}(G) = 1$ and $m_{cc}(G) = 3$. ■

Theorem 3.6. Let $G = C_{p_1} + C_{p_2}$, $p_1 \geq 4,\ p_2 \geq 4$ .Then $m_{cc}(G) = 2$ and

$$f_{mcc}(G) = \begin{cases} & p_1 = 4 \ and \ p_2 = 4 \\ & or \\ 1 & p_1 = 4 \ and \ p_2 = 5 \\ & or \\ & p_1 = 5 \ and \ p_2 = 4 \\ 2 & otherwise \end{cases}$$

Proof. Let C_{p_1} be $v_1, v_2, \ldots\ldots, v_{p_1}$ and C_{p_2} be $u_1, u_2, \ldots\ldots, u_{p_2}$.

Case (i): $p_1 = 4, p_2 = 4$. Then $M_1 = \{v_1, v_3\}, M_2 = \{v_2, v_4\}$, $M_3 = \{u_1, u_3\}$, $M_4 = \{u_2, u_4\}$ are the only four m_{cc}-sets of G such that $f_{mcc}(M_i) = 1 \ to \ 4$ so that and $f_{mcc}(M) = 1$ and $m_{cc}(G) = 2$.

8

Case(ii): $p_1 = 4, p_2 = 5$. Then $M_1 = \{v_1, v_3\}, M_2 = \{v_2, v_4\}$, $M_3 = \{u_1, u_3\}$, $M_4 = \{u_1, u_4\}$, $M_5 = \{u_2, u_4\}, M_6 = \{u_2, u_5\}$, $M_7 = \{u_3, u_5\}$ are the only seven m_{cc}-sets of G

such that $f_{mcc}(M_1) = f_{mcc}(M_2) = 1$ and $f_{mcc}(M_i) = 2$; $i =$ 3 to 7 so that and $f_{mcc}(M) = 1$ and $m_{cc}(G) = 2$.

Case (iii): $p_1 \geq 5, p_2 = 5$. Then $M = \{x, y\}$ be any two non adjacent vertices of C_p and $S = \{u, v\}$ be any two non adjacent vertices of C_{p_2}. Since $p_1 \geq 5, p_2 = 5$, $f_{mcc}(M) = 2$ and $f_{mcc}(G) = 2$ and $m_{cc}(G) = 2$. ■

References:

[1] F.Buckley and F.Harary, *Distance in Graphs*, Addition-Wesley, Redwood City, CA,1990.

[2] Esamel M. Paluga, Sergio R. Canoy, Jr. Monophonic numbers of the join and Composition of connected graphs, *Discrete Mathematics*, 307 (2007), 1146 -1154.

[3] J. John and S. Panchali, The Upper Monophonic Number of a graph, International J. math. Combin. 4(2010), 46-52.

[4] J. John, P. Arul Paul Sudhahar and A. Vijayan, 'The Connected Monophonic Number of a graph', International Journal of Combinatorial Graph Theory and Applications, Vol.,No.1, (January-June 2012), pp 83-90.

[5] J. John and P. Arul Paul Sudhahar On The Edge Monophohic Number of a Graph, Filomat 26:6(2012), 1081-1089.

[6] J. John and S. Panchali, The forcing monophonic number of a graph, International Journal of Mathematical Archive-3(3), 2012, Page:935-938. ISSN 2229-5046.

[7] F. Merlin Sugirtha, P. Arul Paul Sudhahar and S. Robinson Chellathurai, On The Complement Connected Monophonic Number of a Graph, International Journal of Scientific Research and Review, ISSN NO:2279-543X,Vol 7,Issue 11, 2018,225-231.

9