On $r\omega$-closed sets and it's properties in topological spaces

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Abstract. We introduce a new forms of closed sets is regular ω-closed (briefly $r\omega$-closed) in topological spaces. This new class of closed set placed between regular closed sets and g-closed sets. We obtain several characterizations and some of their properties are investigate. Further, we applying $r\omega$-closed sets to introduce a new class of space namely $T_{r\omega}$-space. Also we investigate the relationship between this type of space and other existing spaces on the line of research in this paper.

Keywords and Phrases: r-closed sets, ω-open sets, $T_{\frac{1}{2}}$-spaces and $T_{r\omega}$-spaces.

1. Introduction

In 1970, N. Levine introduced and studied generalized closed (briefly g-closed) sets and semi-open sets respectively. In 1993, Devi.et.al [6] was introduced by the concepts $T\frac{1}{2}$-space, T_b-space and αT_b -space.

We introduce a new forms of closed sets is regular ω-closed (briefly $r\omega$-closed) in topological spaces. This new class of closed set placed between regular closed sets and g-closed sets. We obtain several characterizations and some of their properties are investigate. Further, we applying $r\omega$-closed sets to introduce a new class of space namely $T_{r\omega}$-space. Also we investigate the relationship between this type of space and other existing spaces on the line of research in this paper.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A \subseteq X$, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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Definition 2.1. *A subset A of a space (X, τ) is called*

(i) *α-open [8] if $A \subseteq int(cl(int(A)))$.*
(ii) *semi-open [4] if $A \subseteq cl(int(A))$.*
(iii) *regular open [9] if $A = int(cl(A))$.*
(iv) *pre-open [7] or nearly open [3] if $A \subseteq int(cl(A))$ and pre-closed [7] if $cl(int(A)) \subseteq A$.*

The complements of the above mentioned sets are called their respective closed sets.

Definition 2.2. *A subset A of a (X, τ) is called*

(i) *a generalized closed (briefly g-closed) set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*
(ii) *a generalized semi-closed (briefly gs-closed) set [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*
(iii) *a generalized pre-closed (briefly gp-closed) set [5] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*
(iv) *an a α-generalized closed (briefly a αg-closed) set [6] if an $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*
(v) *ω-closed set [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*

The complements of the above mentioned sets are called their respective open sets.

Definition 2.3. *[1] A space (X, τ) is called*

(i) *$T_{\frac{1}{2}}$-space if every g-closed set is closed.*
(ii) *$T_{\alpha\hat{g}}$-space if every $\alpha\hat{g}$-closed set is α-closed.*
(iii) *αT_b-space if every αg-closed set is closed.*
(iv) *T_b-space if every gs-closed set is closed.*

3. On $r\omega$-closed sets

Definition 3.1. *A subset A of a topological space (X, τ) is said to be Regular ω-closed set(briefly $r\omega$-closed) if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open set.*

The compliment of $r\omega$-closed is called a $r\omega$-open.

Theorem 3.2. *In a topological space (X, τ), every regular closed set is $r\omega$-closed set.*

Proof. Let A be a regular closed and U be any ω-open set containing A in (X, τ). Since A is regular-closed, $cl(int(A)) = A$ For every subset A of X, which implies $rcl(A) = A$. Therefore $rcl(A) \subseteq U$ and hence A is $r\omega$-closed set.

Remark 3.3. *The converse of Theorem 3.2 is not true as seen from the following Example.*

Example 3.4. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{a, c\}, X\}$. In the topological space (X, τ), then the set $\{a, b\}$ is $r\omega$-closed set but not regular-closed.*

Theorem 3.5. *In a topological space (X, τ), every $r\omega$-closed set is g-closed set.*

Proof. Let A be an $r\omega$-closed set and U be any open set containing A in (X, τ). Since every open set is ω-open and A is $r\omega$-closed, $rcl(A) \subseteq U$ for every subset A of X. Since $cl(A) \subseteq rcl(A) \subseteq U$. Which implies $cl(A) \subseteq U$ and hence A is g-closed.

Remark 3.6. *The converse of Theorem 3.5 is not true as seen from the following Example*

Example 3.7. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. In the topological space (X, τ), then the set $\{a, b\}$ is g-closed but not $r\omega$-closed.*

Theorem 3.8. *In a topological space (X, τ), every $r\omega$-closed set is gs-closed.*

Proof. Let A be an $r\omega$-closed and U be any open set containing A in (X, τ). Since every open set is ω-open and A is $r\omega$-closed, $rcl(A) \subseteq U$ for every subset A of X. Since $scl(A) \subseteq rcl(A) \subseteq U$ which impliess $cl(A) \subseteq U$ and hence A is gs-closed.

Remark 3.9. *The converse of Theorem 3.8 is not true as seen from the following Example*

Example 3.10. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. In the topological space (X, τ), then the set $\{a\}$ is gs-closed but not $r\omega$closed.*

Remark 3.11. *In a topological space (X, τ), then the notions of $r\omega$-closed sets and the notions of sg-closed sets are independent of each other as seen from the following Examples.*

Example 3.12. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, X\}$. In a space (X, τ), then the set $\{a, b\}$ is $r\omega$-closed but not sg-closed.*

Example 3.13. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. In the topological space (X, τ), then the set $\{b\}$ is sg-closed but not $r\omega$-closed.*

Theorem 3.14. *In a topological space (X, τ), every $r\omega$-closed set is αg-closed.*

Proof. Let A be an $r\omega$-closed set and U be any open set containing A in (X, τ). Since every open set is ω-open, $rcl(A) \subseteq U$ for every subset A of X. Since $\alpha cl(A) \subseteq rcl(A) \subseteq U$, which implies $\alpha cl(A) \subseteq U$ and hence A is αg-closed.

Remark 3.15. *Converse of the Theorem 3.14 is not true as seen from the following Example.*

Example 3.16. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{b\}, \{a, b\}, X\}$. In a topological space (X, τ), then the set $\{a\}$ is αg-closed but not $r\omega$-closed.*

Remark 3.17. *In a topological space (X, τ), then the notions of $r\omega$-closed sets and $g\alpha$-closed sets are independent of each other as seen from the following Examples.*

Example 3.18. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. In the topological space (X, τ), then the set $\{a, b\}$ is $g\alpha$-closed but not $r\omega$-closed.*

Example 3.19. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{b\}, X\}$. In the topological space (X, τ), then the set $\{a, c\}$ is $r\omega$-closed but not $g\alpha$-closed.*

Theorem 3.20. *In a topological space (X, τ), every $r\omega$-closed set gp-closed.*

Proof. Let A be an $r\omega$-closed and U be any open set containing A in (X, τ). Since every open set is ω-open, $rcl(A) \subseteq U$ for every subset A of X. Since $pcl(A) \subseteq rcl(A) \subseteq U$ which implies $pcl(A) \subseteq U$ and hence A is gp-closed.

Remark 3.21. *Converse of the Theorem 3.20 is not true as seen in the following Example.*

Example 3.22. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a, b\}, X\}$. In the topological space (X, τ), then the set $\{a\}$ is gp-closed but not $r\omega$-closed.*

Remark 3.23. *We obtain the following Diagram-I, where $A \Rightarrow B$ represents A implies B but not conversely.*

$$\begin{array}{ccccc} & & gp\text{-closed set} & & \\ & & \Uparrow & & \\ r\text{-closed set} & \Rightarrow & r\omega\text{-closed set} & \Rightarrow & g\text{-closed set} \\ & & \Downarrow & & \\ & & gs\text{-closed set} & & \end{array}$$

Remark 3.24. *We obtain the following Diagram-II, where $A \nLeftrightarrow B$ represents A and B are independent of each other.*

ω**-closed** $\not\Leftrightarrow$ $r\omega$**-closed** $\not\Leftrightarrow$ α**-closed**

4. Further properties

Theorem 4.1. *In a topological space (X, τ), the union two $r\omega$-closed subsets is $r\omega$-closed.*

Proof. Assuming that A and B are two $r\omega$-closed subset of X. Let U be a ω-open set in X such that $A \cup B \subseteq U$, then $A \subseteq U$ and $B \subseteq U$. Since A and B are $r\omega$-closed subset of X, $rcl(A) \subseteq U$ and $rcl(B) \subseteq U$, but $rcl(A \cup B) \subseteq rcl(A) \cup rcl(B) \subseteq U$. Hence $A \cup B$ is also a $r\omega$-closed subset of X.

Remark 4.2. *In a topological space (X, τ), the intersection of two $r\omega$-closed sets need not to be $r\omega$-closed set as seen from the following Example.*

Example 4.3. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{c\}, X\}$. In a topological space (X, τ), then the sets $C = \{a, c\}$ and $D = \{b, c\}$ are $r\omega$-closed sets but $E = C \cap D = \{c\}$ is not $r\omega$-closed.*

Theorem 4.4. *Let A be a $r\omega$-closed set of (X, τ). Then $rcl(A) - A$ does not contain a non-empty ω-closed set.*

Proof. Suppose that A is $r\omega$-closed. Let G be non-empty ω-closed set contained in $rcl(A) - A$. Now G^c is ω-open set of (X, τ) such that $A \subseteq G^c$. Since A is $r\omega$-closed set of (X, τ), $rcl(A) \subseteq G^c$. Thus $G \subseteq (rcl(A))^c$. Also $G \subseteq rcl(A) - A$. Therefore $G \subseteq (rcl(A))^c \cap rcl(A) = \phi$, Hence $G = \phi$. Which is contradiction to the assumption $G \neq \phi$. Hence $rcl(A) - A$ does not contain a non-empty ω-closed set.

Theorem 4.5. *For A subset of a topological space (X, τ), if A is ω-open and $r\omega$-closed subset then A is a r-closed.*

Proof. Since A is ω-open and$r\omega$-closed subset of (X, τ), $rcl(A) \subseteq A$. Hence A is r-closed.

Theorem 4.6. *In a topological space (X, τ), the intersection of a $r\omega$-closed set and r-closed set is always $r\omega$-closed.*

Proof. Let A be $r\omega$-closed and let F be r-closed. If U is an ω-open set containing $A \cap F$, $A \cap F \subseteq U$, then $A \subseteq U \cup F^c$ and so $rcl(A) \subseteq U \cup F^c$. Now $rcl(A \cap F) \subseteq rcl(A) \cap F \subseteq U$. Hence $A \cap F$ is $r\omega$-closed.

Theorem 4.7. *If A is an $r\omega$-closed set and $A \subseteq B \subseteq rcl(A)$ in a space (X, τ), then B is also a $r\omega$-closed in a space (X, τ).*

Proof. Let U be an ω-open set of (X, τ) such that $B \subseteq U$. Then $A \subseteq U$, since A is $r\omega$-closed set, $rcl(A) \subseteq U$. Also since $B \subseteq rcl(A), rcl(B) \subseteq rcl(rcl(A)) = rcl(A)$. Hence $rcl(B) \subseteq U$. Therefore B is also a $r\omega$-closed set.

5. On some applications

Definition 5.1. *A space (X, τ) is called $T_{r\omega}$-space if every $r\omega$-closed set is r-closed.*

Theorem 5.2. *For a topological space (X, τ) the following conditions are equivalent*

(i) (X, τ) *is a $T_{r\omega}$-space.*

(ii) Every singleton $\{x\}$ is either ω-closed or regular open.

Proof. (1) ⇒ (2) Let $x \in X$, Suppose $\{x\}$ is not a ω-closed set of (X, τ). Then $X - \{x\}$ is not a ω-open set. Thus $X - \{x\}$ is an $r\omega$-closed set of (X, τ). Since (X, τ) is $T_{r\omega}$-space , $X - \{x\}$ is a r-closed set of (X, τ). That is $\{x\}$ is r-open set of (X, τ).

(2) ⇒ (1) Let A be an $r\omega$-closed set of (X, τ). Let $x \in rcl(A)$, By (2) $\{x\}$ is either ω-closed or $r\omega$-open.

Case(a): Let $\{x\}$ be ω-closed. If we assume that $x \in A$, then we would have $x \in rcl(A) - A$. Hence $x \in A$.

Case (b): Let $\{x\}$ be r-open, since $x \in rcl(A)$. $\{x\} \cap A \neq \phi$. This shows that $x \in A$. So in the both cases we have $rcl(A) \subseteq A$. Trivially $A \subseteq rcl(A)$. Therefore $A = rcl(A)$, which implies A is regular-closed. Hence (X, τ) is $T_{r\omega}$-space.

Theorem 5.3. *Every $T_{r\omega}$-space is $T_{\alpha\omega}$-space.*

Proof. : Let (X, τ) be a $T_{r\omega}$-space . Then every singleton is either ω-closed or regular-open. Since every regular-open is α-open, every singleton is either ω-closed or α-open, Hence (X, τ) is a $T_{\alpha\omega}$-space.

Remark 5.4. *The converse of Theorem 5.3 is not true as seen from the following Example.*

Example 5.5. *Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ is $T_{\alpha\omega}$-space but not $T_{r\omega}$-space.*

Remark 5.6. *$T_{\frac{1}{2}}$-space and $T_{r\omega}$-space are independent of one another as seen from the following Examples.*

Example 5.7. *Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Then (X, τ) is a $T_{\frac{1}{2}}$-space but not $T_{r\omega}$-space.*

Example 5.8. *Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then (X, τ) is a $T_{r\omega}$-space but not $T_{\frac{1}{2}}$-space.*

Remark 5.9. *$T_{r\omega}$-space is independent of T_b-space as seen from the following Examples.*

Example 5.10. *Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ) is T_b-space but not $T_{r\omega}$-space.*

Example 5.11. *Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, \{a, c\}, X\}$. Then (X, τ) is $T_{r\omega}$-space but not T_b-space.*

Remark 5.12. *We obtain the following Diagram-III*

(i) Where $A \Rightarrow B$ represents A implies B but not conversely.

(ii) Where $A \not\Leftrightarrow B$ represents A and B are independent.

$$\begin{array}{ccccc} & & T_{\alpha\omega}\textbf{-space} & & \\ & & \Uparrow & & \\ T_{\frac{1}{2}}\textbf{-space} & \not\Leftrightarrow & T_{r\omega}\textbf{-space} & \not\Leftrightarrow & T_b\textbf{-space} \end{array}$$

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