SOME FINITE INTEGRALS INVOLVING I-FUNCTION OF SEVERAL VARIABLES

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Abstract

In aim of this paper is toestablish some finite integrals involving I-Function of several variables.

Keywords: Finite Integral, I-Function of several variables.

1. Introduction:

The I–function of several variables introduced by Sharma and Ahmed given in [1], will be defined and represented as follows:

$$I\,[z_1,\ \ldots,z_r] = I^{0,\ n\ :(m_1,n_1)\ ;\ldots\ldots;\ (m_r,n_r)}_{p_i,q_i:R\ :[p_{i'},q_{i'}:R']\ ;\ldots\ldots;\ [p_{i^{(r)}},q_{i^{(r)}}:R^{(r)}]}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\right|$$

$$\begin{matrix}[(a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,n}],[(a_{ji};\alpha'_{ji},\ldots,\alpha_{ji}^{(r)})_{n+1,p_i}]: \\ [(b_{ji};\beta'_{ji},\ldots,\beta_{ji}^{(r)})_{1,q_i}]:\end{matrix}$$

$$\left.\begin{matrix}[(c'_j,\gamma'_j)_{1,n_1}],[(c'_{ji'},\gamma'_{ji'})_{n_1+1,p_{i'}}];\ \ldots;\ [(c_j^{(r)},\gamma_j^{(r)})_{1,n_r}],[(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}] \\ [(d'_j,\delta'_j)_{1,m_1}],[(d'_{ji'},\delta'_{ji'})_{m_1+1,q_{i'}}];\ \ldots;[(d_j^{(r)},\delta_j^{(r)})_{1,m_r}],[(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}]\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r}\phi_1(\xi_1)\ldots.\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)z_1^{\xi_1}\ldots z_r^{\xi_r}d\xi_1\ldots d\xi_r, \tag{1}$$

where

$\omega = \sqrt{(-1)}$,

$$\phi_k(\xi_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}\xi_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\prod_{j=m_k+1}^{q_i^{(k)}}\Gamma(1-d_{ji}^{(k)}+\delta_{ji}^{(k)}\xi_k)\prod_{j=n_k+1}^{p_i^{(k)}}\Gamma(c_{ji}^{(k)}-\gamma_{ji}^{(k)}\xi_k)]},$$

$\forall k\varepsilon\{1,\ \ldots,\ r\}$;

$$\psi(\xi_1, \ldots, \xi_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} \xi_k)}{\sum_{i=1}^{R} [\prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} \xi_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} \xi_k)}$$

in (1), k in the superscript (k) stands for the number of primes; and $z_1, \ldots, z_r$ are not equal to zero and an empty product is interpreted as unity.

Suppose, as usual, that the parameters

$$\left.\begin{array}{l} a_j, j = 1, \ldots, n;\ a_{ji}, j = n+1, \ldots, p_i;\ b_{ji}, j = 1, \ldots, q_i;\ c_j^{(k)}, j = 1 \ldots, n_k; \\ c_{ji^{(k)}}^{(k)}, j = n_k + 1, \ldots, p_{i^{(k)}}; \\ d_j^{(k)}, j = 1 \ldots, m_k;\ d_{ji^{(k)}}^{(k)}, j = m_k + 1, \ldots, q_{i^{(k)}}; \\ \forall i \in \{1, \ldots, R\}, \forall i^{(k)} \in \{1, \ldots, R^{(k)}\}; \forall k \in \{1, \ldots, r\} \end{array}\right\}$$

are complex numbers, and all the α's, β's, γ's and δ's are assumed to be positive real numbers for standardization purpose such that

$$U_k = \sum_{j=1}^{n} \alpha_j^{(k)} + \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leq 0, \quad (2)$$

$$V_k = \sum_{j=1}^{n} \alpha_j^{(k)} - \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad (3)$$

$\forall i \in \{1, \ldots, R\}, \forall i^{(k)} \in \{1, \ldots, R^{(k)}\}; \forall k \in \{1, \ldots, r\}, p_i, q_i, p_{i^{(k)}}, q_{i^{(k)}}, n, n_{i^{(k)}}$ and $m_{i^{(k)}}, \forall i \in \{1, \ldots, R\}, \forall i^{(k)} \in \{1, \ldots, R^{(k)}\};$ $\forall k \in \{1, \ldots, r\}$ are non-negative integers such that $p_i \geq n \geq 0, p_{i^{(k)}} \geq n_{i^{(k)}} \geq 0, q_i \geq 0, q_{i^{(k)}} \geq 0, \forall i \in \{1, \ldots, R\}, \forall i^{(k)} \in \{1, \ldots, R^{(k)}\}$ and $\forall k \in \{1, \ldots, r\}$.

The definition of the multivariable I-function given by equation (1) will however has a meaning even if some of these quantities are zero. The sequences of parameters in (1) are such that none of the poles of the integrand coincide, that is, the poles of the integrand in (1) are simple. The contour L_k in the complex ξ_k-plane is of the Mellin-Barnes type which runs from −ω∞ to + ω∞ with indentations, if necessary, to ensure that all the poles of $\Gamma\left(d_j^{(k)} - \delta_j^{(k)} \xi_k\right)$, j = 1, …, m_k are separated from those of $\Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)} \xi_k\right)$, j = 1, …, m_k and $\Gamma\left(1 - a_j^{(k)} + \sum_{k=1}^{r} a_j^{(k)} \xi_k\right)$, j = 1, …, n; $\forall k \in [1, \ldots, r]$.

It is known that the multiple Mellin-Barnes contour integral representing the multivariable I-function (1) converges absolutely under the condition (3), when

$$|\arg (z_k)| < \frac{1}{2} V_k \pi, \forall k \in [1, \ldots, r] \quad (4)$$

the point $z_k = 0$; k = 1,…,r and various exceptional parameter values, being tacitly excluded.

Furthermore, we may establish the asymptotic expansions in the following convenient form:

$$I[z_1,\dots,z_r] = \begin{cases} O(|z_1|^{\alpha_1}\dots|z_r|^{\alpha_r}), \max\{|z_1|,\dots,|z_r|\}\to 0 \\ O(|z_1|^{\beta_1}\dots|z_r|^{\beta_r}), n=0, \min\{|z_1|,\dots,|z_r|\}\to\infty \end{cases},$$

where, with k = 1,…,r;

$$\begin{cases} \alpha_k = \min\{Re(d_j^{(k)}/\delta_j^{(k)})\}, j=1,\dots,m_k \\ \beta_k = \max\{Re(c_j^{(k)}-1)/\gamma_j^{(k)}\}, j=1,\dots,n_k \end{cases}$$

provided that each of the inequalities in (2), (3) and (4) holds.

In our investigation we shall need the following results:

From Meulenbeld and Robin [2, p. 343 equ. (38)]:

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P^{m,n}_{k-\left(\frac{m-n}{2}\right)}(x)dx$$

$$=\frac{2^{\rho+\sigma-\left(\frac{m-n}{2}\right)+1}\Gamma\left(\rho-\frac{m}{2}+1\right)\Gamma\left(\sigma+\frac{n}{2}+1\right)}{\Gamma(1-m)\Gamma\left(\rho+\sigma-\frac{m-n}{2}+2\right)}$$

$$.{}_3F_2\left[-k, n-m+k+1, \rho-m/2+1; (1-m), \rho+\sigma-\frac{m-n}{2}+2; 1\right], \quad (5)$$

provided that $Re(\rho-\frac{m}{2}) >-1$, $Re(\sigma+\frac{n}{2}) >-1$.

2. INTEGRALS:

In this section, we shall establish following integrals:

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P^{u,v}_{l-\left(\frac{u-v}{2}\right)}(x)$$

$$I^{0,\ n\ :(m_1,n_1)\ ;\dots\dots;\ (m_r,n_r)}_{p_i,q_i:R\ :[p_{i'},q_{i'}:R']\ ;\dots\dots;\ [p_{i(r)},q_{i(r)}:R^{(r)}]}\left[\begin{matrix} z_1(1-x)^h(1+x)^k \\ \vdots \\ z_r \end{matrix}\right]dx$$

$$=2^{\rho+\sigma-u+v+1}\sum_{t=0}^{\infty}\frac{(-l)_t(v-u+l+1)t}{\Gamma(1-u+t)t!}$$

$$I^{0,\ n\ :(m_1,n_1+2)\ ;\dots\dots;\ (m_r,n_r)}_{p_i,q_i:R\ :[p_{i'}+2,q_{i'}+1:R']\ ;\dots\dots;\ [p_{i(r)},q_{i(r)}:R^{(r)}]}\left[\begin{matrix} z_1 2^{h+k} \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} \dots\dots\dots,\dots\dots:(u-\rho-t,h),(-\sigma-v,k)\dots,\dots,:\dots\dots\dots,\dots\dots \\ \dots\dots\dots,\dots\dots:\dots\dots\dots,(u-v-\rho-\sigma-t-1,h+k):\dots\dots\dots,\dots\dots \end{matrix}\right], \quad (6)$$

provided that h > 0, k > 0 and $|\arg(z_k)| < \frac{1}{2}V_k\pi, k =\in [1,\dots,r]$, where V_k is given in (3).

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P^{u,v}_{l-\left(\frac{u-v}{2}\right)}(x)$$

$$I^{0,\ n\ :(m_1,n_1)\ ;\dots\dots;\ (m_r,n_r)}_{p_i,q_i:R\ :[p_{i'},q_{i'}:R']\ ;\dots\dots;\ [p_{i(r)},q_{i(r)}:R^{(r)}]}\left[\begin{matrix} z_1(1-x)^{-h}(1+x)^k \\ \vdots \\ z_r \end{matrix}\right]dx$$

$$= 2^{\rho+\sigma-u+v+1}\sum_{t=0}^{\infty}\frac{(-l)_t(v-u+l+1)t}{\Gamma(1-u+t)t!}$$

$$I^{0,\ n\ :(m_1+1,n_1+1)\ ;\ldots\ldots;\ (m_r,n_r)}_{p_i,q_i:R\ :[p_{i'}+1,q_{i'}+2:R']\ ;\ldots\ldots;\ [p_{i(r)},q_{i(r)}:R^{(r)}]}\left[\begin{matrix} z_1 2^{-h+k} \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix}\ldots\ldots\ldots,\ldots\ldots:(-\sigma-v,k)\ldots,\ldots,:\ldots\ldots\ldots,\ldots\ldots\\ \ldots\ldots\ldots,\ldots\ldots:(1-u+\rho+t,h),,\ldots\ldots\ldots,(u-v-\rho-\sigma-t-1,-h+k):\ldots\ldots\ldots,\ldots\ldots\end{matrix}\right], \quad (7)$$

provided that h > 0, k > 0 and $|\arg(z_k)| < \frac{1}{2}V_k\pi, \forall k \in [1,\ldots,r]$, where V_k is given in (3).

Proof of (6):

To establish (6), replace the I-function of several variables by its equivalent counter integral as given in (1), we get

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P^{u,v}_{l-\left(\frac{u-v}{2}\right)}(x)$$

$$I^{0,\ n\ :(m_1,n_1)\ ;\ldots\ldots;\ (m_r,n_r)}_{p_i,q_i:R\ :[p_{i'},q_{i'}:R']\ ;\ldots\ldots;\ [p_{i(r)},q_{i(r)}:R^{(r)}]}\left[\begin{matrix} z_1(1-x)^h(1+x)^k \\ \vdots \\ z_r \end{matrix}\right]dx.$$

Interchanging the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, we arrive at

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r}\phi_1(\xi_1)\ldots.\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)z_1^{\xi_1}\ldots z_r^{\xi_r}$$

$$.\left[\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}+h\xi_1}(1+x)^{\sigma+\frac{v}{2}+k\xi_1}P^{u,v}_{l-\left(\frac{u-v}{2}\right)}(x)dx\right]d\xi_1\ldots d\xi_r.$$

On evaluating the inner integral with the help of (5), interpreting the result with the help of (1), the integral (6) is obtained. The results (7) can be established easily in the view of (5) exactly on the same lines as given above.

3. PARTICULAR CASES:

I. On choosing r = 1 and R = r in the integrals (6) and (7), we get following integrals in terms of I-function of one variable:

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P^{u,v}_{l-\left(\frac{u-v}{2}\right)}(x)$$

$$.I^{m,n}_{p_i,q_i:r}[(1-x)^h(1+x)^k z]dx$$

$$= 2^{\rho+\sigma-u+v+1}\sum_{t=0}^{\infty}\frac{(-l)_t(v-u+l+1)t}{\Gamma(1-u+t)t!}$$

$$.\,I^{m,n+2}_{p_i+2,q_i+1:r}\left[z2^{h+k}\middle|^{(u-\rho-t,h),(-\sigma-v,k)\ldots,\ldots,}_{,\ldots\ldots\ldots,(u-v-\rho-\sigma-t-1,h+k)}\right], \quad (8)$$

provided that h > 0, k > 0 and |arg z| < ½πB, where B is given by

$$B = \sum_{j=1}^{npi}\alpha_j - \sum_{j=n+1}^{m}\alpha_{ji} + \sum_{j=1}^{qi}\beta_j - \sum_{j=m+1}\beta_{ji},$$

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P_{l-\left(\frac{u-v}{2}\right)}^{u,v}(x)$$

$$.\mathrm{I}_{\mathrm{p_i,q_i:r}}^{\mathrm{m,n}}[(1-x)^{-h}(1+x)^{k}z]dx$$

$$=2^{\rho+\sigma-u+v+1}\sum_{t=0}^{\infty}\frac{(-l)_t(v-u+l+1)t}{\Gamma(1-u+t)t!}$$

$$.\,\mathrm{I}_{\mathrm{p_i+1,q_i+2:r}}^{\mathrm{m+1,n+1}}\left[\mathrm{z}2^{-\mathrm{h+k}}\Big|_{(1-\mathrm{u}+\rho+\mathrm{t,h}),,\ldots\ldots\ldots,(\mathrm{u}-\mathrm{v}-\rho-\sigma-\mathrm{t}-1,-\mathrm{h+k})}^{(-\sigma-\mathrm{v,k})\ldots,\ldots,,}\right], \quad (9)$$

provided that $h > 0$, $k > 0$ and $|\arg z| < \frac{1}{2}\pi B$, where B is given by

$$B = \sum_{j=1}^{npi}\ \sum_{j=n+1}\alpha_j - \sum_{j=1}^{m}\alpha_{ji} + \sum_{j=1}^{qi}\beta_j - \sum_{j=m+1}\beta_{ji},$$

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