FRACTIONAL DERIVATIVES INVOLVING MULTIVARIABLE G-FUNCTION

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ABSTRACT

The aim of this paper is to obtain some fractional derivatives involving MultivariableG-function.

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1. INTRODUCTION:

The fractional derivatives (and fractional integration) of special functions of one and more variables is important, such as in the evaluation of series and integrals, the derivation of generating functions and the solution of differential and integral equations. Motivated by these and many other avenues of applications, the fractional calculus operator ${}_{\alpha}D_x^{\mu}$is much used in the theory of special functions of one and more variables.

Looking into the requirement and importance of fractional calculus in various branches, in this paper we establish some new fractional derivatives involving Multivariable G-function, which will be useful to analysis the various problems in different fields.

The multivariable G–function given by Khadia and Goyal [1] is defined as follows:

$$G[z_1, \dots, z_r] = G^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;1,\dots,1)_{1,p}: (c_j';1)_{1,p_1};\dots\dots;(c_j^{(r)};1)_{1,p_r} \\ (b_j;1,\dots,1)_{1,q}: (d_j';1)_{1,q_1};\dots\dots;(d_j^{(r)};1)_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)\psi(\xi_1,\dots,\xi_r)z_1^{\xi_1}\dots z_r^{\xi_r}d\xi_1\dots d\xi_r \quad (1)$$

where$\omega = \sqrt{(-1)}$,

$$\psi(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\xi_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\xi_i)\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}\xi_i)}$$

$$\phi_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\xi_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\xi_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\xi_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\xi_i)}$$

In (1), i in the superscript (i) stands for the number of primes, e.g., $b^{(1)} = b'$, $b^{(2)} = b''$, and so on; and an empty product is interpreted as unity.

Suppose, as usual, that the parameters

a_j, j = 1, …., p; $c_j^{(i)}$, j = 1,….,p_i;

b_j, j = 1, …., q; $d_j^{(i)}$, j = 1,….,q_i; $\forall i \in \{1,\ldots,r\}$

are complex numbers. Also

$|\arg(z_i)| < \frac{1}{2}V_i\pi, \forall i \in [1,\ldots,r]$, where V_i is

$$V_i = n + \sum_{i=1}^{r} m_i + \sum_{i=1}^{r} n_i - \frac{1}{2}\left(p + q + \sum_{i=1}^{r} p_i + \sum_{i=1}^{r} q_i\right) \qquad (2)$$

where the integral n, p, q, m_i, n_i, p_i and q_i are constrained by the inequalities $p \geq n \geq 0$, $q \geq 0$, $q_i \geq m_i \geq 1$ and $p_i \geq n_i \geq 1$ $\forall i \in \{1, 2, \ldots, r)$ and the inequalities in (2) hold for suitably restricted values of the complex variables $z_1, \ldots, z_r$. The sequence of parameters in (1) are such that none of the poles of the integrand coincide, that is, the poles of the integrand in (1) are simple. The contour L_i in the complex ξ_i–plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)}-\xi_i)$, j = 1,…, m_i are separated from those of $\Gamma(1 - c_j^{(i)}+\xi_i)$, i = 1, …., n_i.

In the present investigation we require the following formula:

From Rainvile [3, p.12]:

$$\Gamma(z+1) = z\Gamma(z). \qquad (3)$$

2. FRACTIONAL DERIVATIVES:

In this section, we establish some fractional derivatives involving multivariable G-function as follows:

$$D_x^{\mu}\{x^k(x^{\nu_1}+a)^{\lambda}(b-x^{\nu_2})^{-\delta} \times G^{0,\ n\ :(m_1,n_1);\ldots\ldots;(m_r,n_r)}_{p,q\ :(p_1,q_1);\ldots\ldots;(p_r,q_r)}\left[\begin{matrix} z_1 x(x^{\nu_1}+a)^{-1}(b-x^{\nu_2})^{-1} \\ \vdots \\ z_r \end{matrix}\right]\}$$

$$= a^{\lambda}b^{-\delta}x^{k-\mu}\sum_{r,s=0}^{\infty}\frac{(x^{\nu_1}/a)^r(x^{\nu_2}/b)^s}{r!\,s!} \times G^{0,\ n\ :(m_1+1,n_1+2);\ldots\ldots;(m_r,n_r)}_{p,q\ :(p_1+3,q_1+3);\ldots\ldots;(p_r,q_r)}\left[\begin{matrix} z_1 x a^{-1} b^{-1} \\ \vdots \\ z_r \end{matrix}\right|$$

$$\left.{}^{\ldots\ldots\ldots\ldots,\ldots\ldots\ldots:(1-\delta-s,1),(-k-r\nu_1-s\nu_2,1),\ldots\ldots\ldots\ldots,(1+\lambda-r,1):\ldots\ldots\ldots,\ldots\ldots\ldots}_{\ldots\ldots\ldots\ldots,\ldots\ldots\ldots:(1+\lambda,1),\ldots\ldots\ldots,(1-\delta,1),(-k+\mu-r\nu_1-s\nu_2,1):\ldots\ldots\ldots,\ldots\ldots\ldots}\right], \quad (4)$$

provided (in addition to the appropriate convergence and existence conditions) that min$\{\nu_1, \nu_2\} > 0$, $|\arg(z_i)| < \frac{1}{2}V_i\pi, \forall i \in [1, \ldots, r]$, where V_i is given in (2).

$$D_x^{\mu}\{x^k(x^{\nu_1}+a)^{\lambda}(b-x^{\nu_2})^{-\delta}$$

$$\times\, G^{0,\ n\ :(m_1,n_1);\ldots\ldots\ldots;\ (m_r,n_r)}_{p,q\ :(p_1,q_1)\ ;\ldots\ldots\ldots;\ (p_r,q_r)}\left[\begin{matrix} z_1x^{-1}(x^{\nu_1}+a)(b-x^{\nu_2})^{-1} \\ \vdots \\ z_r \end{matrix}\right]\}$$

$$= a^{\lambda}b^{-\delta}x^{k-\mu}\sum_{r,s=0}^{\infty}\frac{(x^{\nu_1}/a)^r(x^{\nu_2}/b)^s}{r!\ s!}\times$$

$$G^{0,\ n\ :(m_1+1,n_1+2);\ldots\ldots\ldots;\ (m_r,n_r)}_{p,q\ :(p_1+3,q_1+3)\ ;\ldots\ldots\ldots;\ (p_r,q_r)}\left[\begin{matrix} z_1x^{-1}ab^{-1} \\ \vdots \\ z_r \end{matrix}\right|$$

$$\left.{}^{\ldots\ldots\ldots\ldots,\ldots\ldots\ldots:(-\lambda,1),(1-\delta-s,1),\ldots\ldots\ldots\ldots,(1+k-\mu+r\nu_1+s\nu_2,1):\ldots\ldots\ldots,\ldots\ldots\ldots}_{\ldots\ldots\ldots\ldots,\ldots\ldots\ldots:(1+k+r\nu_1+s\nu_2,1),\ldots\ldots\ldots,(-\lambda+r,1),(1-\delta,1):\ldots\ldots\ldots,\ldots\ldots\ldots}\right], \quad (5)$$

provided (in addition to the appropriate convergence and existence conditions) that min$\{\nu_1, \nu_2\} > 0$, $|\arg(z_i)| < \frac{1}{2}V_i\pi, \forall i \in [1, \ldots, r]$, where V_i is given in (2).

Proof:

To prove of (4), we first replace the multivariable G-function occurring on the L.H.S. by its Mellin-Barnes contour integrals, collected the powers of x, $(x^{\nu_1}+a)$ and $(b-x^{\nu_2})$ and apply binomial expansion:

$$(x+\xi)^{\lambda} = \xi^{\lambda}\sum_{r=0}^{\infty}\binom{\lambda}{r}\left(\frac{x}{\xi}\right)^r;\ \left|\frac{x}{\xi}\right| < 1, \quad (6)$$

we then apply the formula [2, p.67 eq.(4.4.4)]:

$$D_x^{\mu}(x^{\lambda}) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)}.x^{\lambda-\mu};\ (\mathrm{Re}(\lambda) > -1), \quad (7)$$

and interpret the resulting Mellin-Barnes contour integrals as multivariable G-function, we shall arrive at (4).

Preceding on similar lines the results from (5) can be derived easily.

3. PARTICULAR CASES:

On specializing the parameters in (4) and (5), we get following fractional derivatives in terms of G-function of one variable:

$$D_x^{\mu}\{x^k(x^{\nu_1}+a)^{\lambda}(b-x^{\nu_2})^{-\delta}\ G[zx(x^{\nu_1}+a)^{-1}(b-x^{\nu_2})^{-1}]\}$$

$$= a^{\lambda}b^{-\delta}x^{k-\mu}\sum_{r,s=0}^{\infty}\frac{(x^{\nu_1}/a)^r(x^{\nu_2}/b)^s}{r!\ s!}\times$$

$$G^{m+1,n+2}_{p+3,q+3}\left[zxa^{-1}b^{-1}\left|^{(1-\delta-s,1),(-k-r\nu_1-s\nu_2,1),(a_j,1)_{1,p}(1+\lambda-r,1)}_{(1+\lambda,1),(b_j,1)_{1,q},(1-\delta,1),(-k+\mu-r\nu_1-s\nu_2,1)}\right.\right], \quad (8)$$

provided (in addition to the appropriate convergence and existence conditions) that min$\{\nu_1, \nu_2\} > 0$, $|\arg z| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$.

$$D_x^{\mu}\{x^k(x^{\nu_1}+a)^{\lambda}(b-x^{\nu_2})^{-\delta}\, G[zx^{-1}(x^{\nu_1}+a)(b-x^{\nu_2})^{-1}]\}$$

$$= a^{\lambda}b^{-\delta}x^{k-\mu}\sum_{r,s=0}^{\infty}\frac{(x^{\nu_1}/a)^r(x^{\nu_2}/b)^s}{r!\,s!}\times$$

$$G_{p+3,q+3}^{m+1,n+2}\left[zx^{-1}ab^{-1}\middle|_{(1+k+r\nu_1+s\nu_2,1),(b_j,1)_{1,q},(-\lambda+r,1),(1-\delta,1)}^{(-\lambda,1),(1-\delta-s,1),(a_j,1)_{1,p},(1+k-\mu+r\nu_1+s\nu_2,1)}\right], \qquad (9)$$

provided (in addition to the appropriate convergence and existence conditions) that $\min\{\nu_1, \nu_2\} > 0$, $|\arg z| < (m + n - ½\, p - ½\, q)\pi$.

REFERENCES

1. Khadia, S. S. and Goyal, A. N.: On the generalized function of 'n' variables, VijnanaParishadAnusandhanPatrika, 13(1970), 191-201.
2. Oldham, K. B. and Spanier, J.: The Fractional Calculas, Academic Press, NY/London, (1974).
3. Rainville, E. D.: Special Functions, Macmillan, NewYork, 1960.