On the System of Equations $x+y=z+w,\ y+z=(x+w)^3$

A. Vijayasankar[#1], M.A. Gopalan*[2], Sharadha Kumar[#3]

[1#] *Assistant Professor, Department of Mathematics, National College, Trichy-620 001, Tamil Nadu, India.*
[2*] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India.
[3#] *Research Scholar, Department of Mathematics, National College, Trichy-620 001, Tamil Nadu, India.*
.[1] avsankar70@yahoo.com
[2] mayilgopalan@gmail.com
[3] sharadhak12@gmail.com

***Abstract*—This paper concerns with the problem of obtaining non-zero distinct integer solutions to the system of equations $x+y=z+w$, $y+z=(x+w)^3$.**

***Keywords*— System of double equations, integer solutions, diophantine 3-tuples, dio 3-tuples.**

I. INTRODUCTION

Number Theory has occupied a significant position in the world of Mathematics. One of the enjoyable areas of Number Theory that has not only attracted but also motivated many Mathematicians since antiquity is the subject of patterns in numbers. Mans love for numbers is perhaps older than Number Theory. Nearly, ever century has witnessed new and fascinating discoveries about the properties of numbers [1-5]. They form sequences, they form patterns and so on. Numerous discoveries arise from these peculiar number patterns.

Now, consider the positive integers 2, 4, 107, 109. Note that, $4+107=2+109$ and $107+109=(2+4)^3$. This illustration motivated us for searching non-zero distinct integer quadruples (x,y,z,w) such that, $x+y=z+w$, $y+z=(x+w)^3$. A few interesting properties among the solutions are presented. Sequences of diophantine 3-tuples with suitable properties are exhibited.

II. DEFINITIONS

Diophantine 3-tuple: A triple (a,b,c) is said to be a Diophantine 3-tuple, if the product of any two members of the set added with non-zero integer or a polynomial is a perfect square.

Dio 3-tuple: A triple (a,b,c) is said to be a Dio 3-tuple, if the product of any two members of the set added with the same members and increased by non-zero integer or a polynomial is a perfect square.

III. METHOD OF ANALYSIS

This paper illustrates the process for obtaining non-zero distinct integer solutions to the pair of equations

$$x+y=z+w \tag{1}$$

$$y+z=(x+w)^3 \tag{2}$$

Consider the linear transformations

$$x=u+v,\ w=u-v,\ u\neq v\neq 0 \tag{3}$$

Substituting (3) in (1) and (2) and simplifying, we have,

$$z = 4u^3 + v, y = 4u^3 - v \quad (4)$$

Note that (3) and (4) satisfy (1) and (2).

A few interesting relations observed among the solutions are as follows:

I. Each of the following expressions represents a cubical integer.
   (i). $2z + w - x$
   (ii). $2y + x - w$

II. $x + z \equiv (y + w)(\mathrm{mod}\, 4)$

III. Each of the following triples represents Pythagorean triples
   (i). $(4\alpha^6 v^3 - v, 4\alpha^3 v^2, 4\alpha^6 v^3 + v)$
   (ii). $(12\alpha^6 v^3 + 8\alpha^3 v^2 + v, 16\alpha^6 v^3 + 4\alpha^3 v^2, 20\alpha^6 v^3 + 8\alpha^3 v^2 + v)$

IV. The following Table 1 represents Diophantine 3-tuples with suitable property

Table 1: Diophantine 3-tuples

Property	***3-tuples***
$D(v^2 + 2nu + n^2)$	1. $(x, c_s, c_{s+1}), c_s = (s+1)^2 u + (s^2 - 1) v + 2sn, s = 1,2,3.......$ 2. $(w, c_s, c_{s+1}), c_s = (s+1)^2 u - (s^2 - 1) v + 2sn, s = 1,2,3.......$
$D(v^2 + 8nu^3 + n^2)$	1. $(y, c_s, c_{s+1}), c_s = 4(s+1)^2 u^3 - (s^2 - 1) v + 2sn, s = 1,2,3.......$ 2. $(z, c_s, c_{s+1}), c_s = 4(s+1)^2 u^3 + (s^2 - 1) v + 2sn, s = 1,2,3.......$

V. The following Table 2 represents Dio 3-tuples with suitable property

Table 2: Dio 3-tuples

Property	***3-tuples***
$D(v^2 + (2s - 2)u + s^2)$	$(u + v, u - v, 4u + 2s + 1)$
$D(v^2 + s^2 + (8s - 8)u^3)$	$(4u^3 + v, 4u^3 - v, 16u^3 + 2s + 1)$

VI. It is worth to note that, each of the values of $y + x, z + w$ represents centered triangular pyramidal number.

VII. The values of $(y + z) - (x + w)$ represents six times centered octagonal pyramidal number.

VIII. The values of $x + y + z + w$ represents two times centered triangular pyramidal number.

IX. $2\left(z^2+y^2\right)-(x-w)^2$ is a sextic integer.

X. $2(xy+zw)+(z-y)^2$ is a bi-quadratic integer.

XI. $2(xy-zw)=\left(x^2-w^2\right)\left((x+w)^2-1\right)$

XII. When $u=3r^2-s^2,\ v=2rs,\ r\neq s\neq 0$

x^2-xw+w^2 is a perfect square.

XIII. Consider, u,v to represent the sum of the legs and difference between the legs of the Pythagorean triangle then, it is observed that x^2+w^2 is a perfect square.

IV. Conclusions

In this paper, an attempt has been made to obtain many integer solutions to the pair of equations $x+y=z+w$, $y+z=(x+w)^3$. The authors wish that the researchers of diophantine equations maybe motivated in solving other choices of double diophantine equations.

References

[1] L.E. Dickson, History of the theory of Numbers, Vol.II, Chelsea publishing company, New York, 1952.

[2] B. Batta and A.N. Singh, History of Hindu Mathematics, Asia Publishing House, 1938.

[3] Georges Ifrah, The Universal History of Numbers, The Harvill Press, London (1998).

[4] Shailesh Shirali, Number Sequences, Universities Press (2001).

[5] J.H.Conway, Richard K Guy, The Book of Numbers, Springer-verlog, New York (1996).